A STUDY OF A CLASS OF NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS

BY

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DEPARTMENT OF MATHEMATICS
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TO THE MEMORY OF MY MOTHER

CERTIFICATE

Certified that the thesis entitled "A Study of a Class of Neutral Functional Differential Equations" by Narahari Parhi has been carried out under my supervision and this work has not been submitted elsewhere for a degree.

PCDas

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May - 1971

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This thesis has [illegible]

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Narahari Parhi

NARAHARI PARHI

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CONTENTS

SYNOPSIS

Differential equations with delayed arguments, differential equations with advanced arguments and neutral differential equations go under the name of differential equations with deviating arguments or more generally functional differential equations Out of these three types of equations, differential equations with delayed arguments are most important due to the frequent occurance of these equations in applied science and technology. Although the rate of occurance is less in case of neutral differential equations still these equations do have practical importance which may be realised from the work of R.D Driver and R K. Brayton [''A functional differential system of neutral type arising in a two body problem of classical electrodynamics'' Int. Symp. nonlinear diff. equa. and nonlinear mech., Ed. J.P. LaSalle and S. Lefschetz, A.P., 1963, pp 474 and ''Bifurcation of periodic solutions in a nonlinear differential equation of neutral type'' Quart. Appl. Math , 24(1966), 215-224]

The theory of differential equations with delayed arguments is well established. But much is not known about neutral differential equations. This is due to, among other things, lack of a suitable metric. However, in recent years, some results were obtained by G.A Kamenskii, R.D Driver, J.K Hale, K.R. Meyer and M A. Cruz.

All but Kamenskii studied a class of neutral differential equations in which the derivative with delayed argument occurs linearly. The equation studied by Hale and Cruz is more general than the one studied by Driver But they used different metrics Hale and Cruz, in their work, covered problems like the existence, uniqueness, continuous dependence of solutions, variation of constants formula and stability of solutions.

In this work, in contradistinction to that of Hale and Cruz, we consider a class of neutral differential equations in which the derivative with delayed argument does not necessarily occur in a linear fashion. But this work neither includes nor is included in the work of above mentioned authors For our work we use the well-known bounded variation norm for the space of absolutely continuous functions.

The present thesis is divided into five chapters. The first chapter deals with the existence, uniqueness, extended existence and continuous dependence on the initial data, right-hand side of the equation and parameters of solutions of the neutral differential equations of the type

(*) $$\frac{dx}{dt} = f(t, x_t, x_t)$$

where $f: (a,b) \times D \longrightarrow E^n$ and D is an open set in $AC(I_o)$, the space of absolutely contiruous functions ψ $[-\tau, 0] \longrightarrow E^n$ Here x_t $[-\tau, 0] \longrightarrow E^n$ defined by $x_t(\theta) = x(t+\theta)$, $t \in (a,b)$ and $x \in AC((a-\tau, b))$. In this case it is postulated that

$f(t,x_t,\dot{x}_t)$ satisfies, besides other conditions, Lipschitz condition.

In Chapter 2 an attempt has been made to relax the Lipschitz condition on the right-hand side of the equation (*) by some other regularity condition This enables us to establish simple existence of solution of (*) under more general conditions Precisely speaking, the Lipschitz condition is not implied by the regularity condition, which is illustrated by an example, but the existence theorem in Chapter 2 establishes, as a corollary, the existence of solutions of the autonomous equation

$$(**) \qquad \frac{dx}{dt} = f(x_t, x_t)$$

under conditions of the corresponding theorem of Chapter 1 A theorem concerning the global existence of solutions of (*) is also given here.

In third chapter we deal with the equation (*) where $f(t,x_t,x_t)$ is linear in x An existence theorem for the solution of such an equation and a representation of solutions has been given. Some theorems concerning the variational system are also obtained.

Chapter 4 concerns periodic solutions. Theorems establishing the existence of periodic solutions of the linear system (*) and the quasi-linear system

$$\frac{dx}{dt} = f(t,x_t,\dot{x}_t) + h(t,x_t,\dot{x}_t,\mu)$$

have been proved for small μ .

The stability theory has been discussed in the last chapter. Two types of stability are defined and theorems on stability, uniform stability, uniform asymptotic stability, exponential stability have been proved with the help of Liapunov functionals. Also some of the converse theorems have been obtained

CHAPTER 0

INTRODUCTION

0.1 CLASSIFICATION

Differential equations with deviating arguments were encountered in the work of L.Euler, but these equations were studied systematically in the twentieth century to meet the needs of the applied science and technology, specially of control theory.

In the last two decades the area of application of differential equations with deviating arguments has greatly expanded and such applications are not only felt in physics and technology but also in some areas of economics and biological sciences. This abundance of applications has increased the interest in the theory of differential equations with deviating arguments and a lot of papers, in mathematical literature, have been devoted to such equations - particularly in Soviet Union and United States (see bibliographies in [2], [10] and [11]).

Differential equations with deviating arguments are differential equations in which the unknown function enters with several different values of the argument. The following shows the variety of such equations:

$$(0.1.1) \qquad x(t) = f(t, x(t), x(t-\tau)),$$

$$(0\ 1\ 2) \qquad x(t) = f(t, x(t), x(t-\tau_1), x(t-\tau_2)),$$

$$(0\ 1.3) \qquad \ddot{x}(t) = f(t, x(t), x(t-\tau(t)), x(t-\tau(t)))$$

$$(0.1\ 4) \qquad x(t) = f(t, x(t), x(t/2), x(t^2))$$

$$(0.1\ 5) \qquad x(t+\tau) = f(t, x(t), x(t+\tau), x(t), x(t+\tau))$$

$$(0\ 1\ 6) \qquad \dot{x}(t) = f(t, x(t), x(t/2), x(t-e^{-t}))$$

$$(0.1\ 7) \qquad \dot{x}(t) = f(t, x(t), x(t-\tau), \dot{x}(t-\tau))$$

$$(0\ 1\ 8) \qquad \dot{x}(t) = f(t, x(t), x(t-\tau_1), x(t-\tau_2))$$

$$(0\ 1\ 9) \qquad x(t) = f(t, x(t-\tau), x(t), x(t+\tau))$$

$$(0\ 1.10) \qquad \dot{x}(t) = f(t, x(t), x(t-\tau(t)), \dot{x}(t-\tau(t)))$$

$$(0\ 1.11) \qquad \dot{x}(t) = f(t, x(t), x(t-\tau(t)), x(+-\tau(t)), \ddot{x}(t-\tau(t))).$$

Differential equations with retarded arguments are differential equations with deviating arguments in which the highest order derivatives of the unknown function appear for just one value of the argument and this argument is greater than or equal to all arguments of the unknown function and its derivatives appearing in the equation. For example, equations (0.1.1), (0.1.2), (0.1.3), (0.1 4), (0.1.5), (0.1.6) and (0.1.11) are equations with retarded arguments if $\tau>0$ in (0.1.1), $\tau_1>0$ and $\tau_2>0$ in (0.1.2), $\tau(t)>0$ in (0.1.3), $0<t<1$ in (0.1.4) and so on

Differential equations with advanced arguments are those with deviating arguments in which the highest order derivatives of the unknown function appear for just one value of the argument and this argument is less than or equal to all arguments of the unknown function and its derivatives appearing in the equation. For example (0.1.1),(0.1.2),(0.1.3)

are equations with advanced arguments if $\tau<0$ in (0 1.1), $\tau_1<0$ and $\tau_2<0$ in (0.1.2) and $\tau(t)<0$ in (0.1.3).

All remaining types of differential equations with deviating arguments are called differential equations of neutral type. For example, equation (0 1 4) for $t>1$ with $\frac{\partial f}{\partial x(t/2)} \neq 0$ and $\frac{\partial f}{\partial x(t^2)} \neq 0$, (0 1.7) when $\tau>0$ and $\frac{\partial f}{\partial \dot{x}(t-\tau)} \neq 0$ and (0.1 9) with $\frac{\partial f}{\partial x(t-\tau)} \neq 0$ and $\frac{\partial f}{\partial x(t+\tau)} \neq 0$

It is interesting to note that an equation may belong to one of the above mentioned types on one set of values of t and to another type on another set. A typical example of such an equation is the equation (0.1.4) which, for $0<t<1$, is of retarded type, for $t<0$, is of advanced type and, for $t>1$, is of neutral type.

An analogous classification also valid for systems of differential equations. In that case it is sufficient to change the word ''function'' to ''vector function'' in all definitions.

Equations with retarded arguments and equations of neutral type are encountered in many applications. But equations of advanced type are rarely encountered. Neutral equations and equations with retarded arguments describe many physical and sociological processes with ''after effects''

Differential equations with retarded arguments are also some times called delay differential equations or hystero-differential equations. Yet another terminology used in this connection is ''differential difference equations''.

It is quite possible that the delays in above mentioned equations not only may depend on the independent variable 't' but also on the unknown function as in the following examples

$$(0.1.12) \qquad \dot{x}(t) = f(t,x(t),x(t-\tau(t,x(t))))$$

$$(0.1.13) \qquad x(t) = f(t,x(t),x(t-\tau_1(t,x(t))),\dot{x}(t-\tau_2(t,x(t)))).$$

In this work we shall be concerned with those differential equations of neutral type which are often encountered in applications. Since t usually stands for time in applications, we shall be interested in continuing solutions in the direction of increasing t.

The simplest type of differential equation of neutral type is given by the equation (0.1 7). In a more general set up, we consider the following equation:

$$(0.1.14) \qquad \dot{x}(t) = f(t,x_t,\dot{x}_t) \ ,$$

where $f\ (a,b) \times D \longrightarrow E^n$, E^n is n-dimensional Euclidean space, (a,b) is an interval in the real line and D is an open set in $AC([-\tau,0],E^n)$, the space of absolutely continuous functions $\psi:[-\tau,0] \longrightarrow E^n$ Here $x_t\ \ [-\tau,0] \rightarrow E^n$ is defined by $x_t(\theta) = x(t+\theta)$ for $t \in (a,b)$ and an absolutely continuous function $x\ (a-\tau,b) \longrightarrow E^n$ Thus the function f depends on whole behaviour of function x and its derivative on $[t-\tau,t]$. Equations of the form (0.1.14) are called functional differential equations of neutral type or neutral functional differential equations (FDENT)

In the following we formulate the initial value problem for the system (0.1 14). Let $t_0 \in (a,b)$ and $\phi \in D$. The initial value problem consists in determining an absolutely continuous function $x\ [t_0-\tau, t_0+\alpha] \longrightarrow E^n$, $\alpha>0$, $t_0+\alpha<b$, such that (i) $x_t \in D$, for $t\in[t_0,t_0+\alpha]$, (ii) $x_{t_0} = \phi$ and (iii) $x(t)$ satisfies (0 1 14), for $t\in[t_0,t_0+\alpha]$. In that case ϕ is called an initial function and $x(t)$ is said to be a solution of (0.1 14) in $[t_0,t_0+\alpha]$

Equation (0.1 14) includes the following equations

$$(0.1\ 15)\qquad x(t) = \int_{-\tau}^{0} K(t,\theta)x_t(\theta)d\theta + \int_{-\tau}^{0} K^*(t,\theta)\dot{x}_t(\theta)d\theta$$

$$(0\ 1\ 16)\qquad x(t) = [\int_{-\tau}^{0} K(t,\theta)x_t(\theta)d\theta]^2 + [\int_{-\tau}^{0} K^*(t,\theta)\dot{x}_t(\theta)d\theta]^2$$

$$(0.1\ 17)\qquad x(t) = \int_{-\tau}^{0} d_\theta\, \eta(t,\theta)x_t(\theta) + \int_{-\tau}^{0} K_1(t,\theta)\dot{x}_t(\theta)d\theta$$

$$(0\ 1.18)\qquad x(t) = [\int_{-\tau}^{0} d_\theta\, \eta(t,\theta)x_t(\theta)]^2 + [\int_{-\tau}^{0} K_1(t,\theta)\dot{x}_t(\theta)d\theta]^2$$

It also includes the following as particular cases

$$(0.1.19)\qquad \dot{x}(t) = \int_{-\tau}^{0} d_\theta \eta(t,\theta)x_t(\theta) + \int_{-\tau}^{0} d_\theta\, \eta^*(t,\theta)\dot{x}_t(\theta)$$

$$(0.1.20)\qquad \dot{x}(t) = x_t(\theta)$$

But the study in this work is not applicable to systems (0.1.19) and (0 1.20)

0.2 HISTORICAL NOTES

Differential equations with deviating arguments have been extensively studied by El'sgol'ts [10,11], Bellman and

Cooke [2] , Myshkis [29,30], Pinney [32] , Krasovskii [26] , Shimanov [34,35,36] and many others For complete bibliography one may see Zverkin, Kamenskii, Norkin, El'sgol'ts [42,43,44] The functional differential equations (FDE), a generalization of the differential-difference equations, were first systematically studied by Krasovskii in his monograph [26] In recent years a lot of efforts have been devoted to FDE Hale [14-18] , Halanay [12] , Krasovskii [26] and Shimanov [34,35,36] made important contributions to the theory of FDE Oguztoreli [31] has studied FDE with infinite delays Many results from the theory of ordinary differential equations have been translated to the general set up of FDE which enables one to construct a fairly general theory of such equations. Differential equations of neutral and advanced types are least studied.

The theory of differential equations of neutral type is not well developed despite the interest of many authors. In simplest equations of neutral type, the first difficulty arises because of the occurance of the derivative of unknown function with a retardation. This leaves much freedom in the choice of the metric on the solution space as well as on the space of initial conditions. The metric must be chosen in a way so as to obtain solutions which are at least continuous with respect to the initial data.

Wright [39] and Bellman and Cooke [2] have considered neutral differential equations in case of constant delays. Infact Bellman and Cooke, in their excellent book [2] , have

discussed the existence and uniqueness of solutions of first order linear neutral differential equations with constant coefficients of the following type

$$(0.2.1)\quad a_0\dot{x}(t)+a_1\dot{x}(t-\tau)+b_0x(t)+b_1x(t-\tau) = h(t),\ a_0\neq 0, a_1\neq 0$$

Exponential bounds and series expansions of solutions of (0.2.1) also have been obtained. They and Brumley [4] have studied the asymptotic behaviour and stability of solutions of homogeneous equations of neutral type

$$(0.2.2)\quad a_0\dot{x}(t)+a_1\dot{x}(t-\tau)+b_0x(t)+b_1x(t-\tau)=0,\ a_0a_1\neq 0$$

Most of these results have been extended, in general, to the following system of linear neutral differential equations with variable coefficients

$$(0.2.3)\quad \dot{z}(t+\tau_m) + \sum_{n=0}^{m-1} A_n(t)\dot{z}(t+\tau_n) + \sum_{n=0}^{m} B_n(t)z(t+\tau_n)=h(t),$$

$t>t_0$, where $A_n(t)$ and $B_n(t)$ are given NxN matrices ($n=0,1,\ldots m$), $h(t)$ is a given column vector of N dimensions and $\tau_0<\tau_1<\tau_2<\ldots<\tau_m$, and, in particular, to

$$(0.2.4)\quad \dot{x}(t+\tau)+a(t)\dot{x}(t)+b(t)x(t)+c(t)x(t+\tau) = h(t).$$

El'sgol'ts [10,11], Kamenskii [23,24] and Driver [7,8] have studied neutral differential equations with variable delays. In [23,24], Kamenskii has considered the space of once continuously differentiable functions and used the following metric.

$$||x-y||_{[a,b]} = \sup_{t\in[a,b]} |x(t)-y(t)| + \sup_{t\in[a,b]} |\dot{x}(t)-\dot{y}(t)|$$

In [7], Driver has indicated the shortcomings of the above metric and consequently used the metric

$$||x-y||_{[a,b]} = \sup_{t\in[a,b]} |x(t)-y(t)| + \int_a^b |\dot{x}(t)-\dot{y}(t)|\,dt$$

in the space of absolutely continuous functions. He established theorems concerning the local and extended existence, uniqueness and continuous dependence of solutions for the following neutral differential equations

(0.2 5) $\dot{x}(t)=f(t,x(t),x(g_1(t)), \quad ,x(g_m(t)),\dot{x}(h_1(t)), \quad ,\dot{x}(h_p(t))),$

$t>t_0$, where $x=(x_1,\ldots,x_n)$ and $f=(f_1, \quad ,f_n), g_1, \quad ,g_m$ and $h_1, \quad ,h_p$ were given functions such that, for some $\alpha<t_0, \alpha\leq g_j(t)\leq t$ $(j=1, \quad .,m)$ and $\alpha\leq h_k(t)<t, (k=1, \; . \; ,p)$ for $t\geq t_0$ and where $\dot{x}(h_k(t))$ represents $\frac{dx(s)}{ds}$ evaluated at $s=h_k(t)$. In case $\alpha= - \infty$, all intervals of the form $[\alpha,$ should be replaced by $(- \infty,$ . .Precisely speaking, Driver has proved theorems concerning the extended existence and uniqueness of solutions of the system (0.2.5) (Theorems 1 and 3 in [7]) where he assumed, besides other conditions, the initial function ϕ to be absolutely continuous with the property that $\dot{\phi}(t)$ is bounded for almost all t in $[\alpha,t_0]$. Further he established Theorems (Theorem 2,4 and 5 and example 5 in [7]) on the extended existence, uniqueness, global existence and continuous dependence on the initial data and right-hand

side of the equation of solutions of the following system:

(0.2 6) $\dot{x}(t) = f^{(1)}(t,x(t),x(g(t)))+f^{(2)}(t,x(t),x(g(t)))\dot{x}(h(t))$,

$t>t_0$. Here $g(t)=(g_1(t), \ldots, g_m(t)), h(t)=(h_1(t), \quad ,h_p(t))$, and the restriction on $\phi(t)$ is dropped

Hale and Meyer [19] have investigated a class of functional integral equations of the type

$$(0.2.7)\qquad \begin{aligned} &x_{t_0} = \phi \\ &x(t)=\gamma(t_0,\phi)+g(t,x_t)+\int_{t_0}^{t} f(s,x_s)ds+\int_{t_0}^{t} h(s)ds, t\geq t_0, \end{aligned}$$

where $t_0\in[\sigma,\infty)$, $\phi\in C \equiv C([-\tau,0],E^n)$, the space of continuous functions mapping $[-\tau,0]$ into E^n, g and f are continuous functions from $[\sigma,\infty) \times C$ into E^n such that, for each $t\in[\sigma,\infty)$, $f(t,\ )$ and $g(t,\ )$ are linear operators, $h\in L_1([t_0,\infty),E^n)$ and $\gamma(t_0,\phi) = \phi(0)-g(t_0,\phi)$. Equation (0.2.7) includes certain classes of neutral equations They established theorems concerning the existence and uniqueness of solutions of (0.2.7) and obtained variation of constants formula. Further they investigated autonomous, homogeneous equations

$$(0.2.8)\qquad \begin{aligned} &x_0= \phi \\ &x(t)= \gamma(\phi)+g(x_t)+ \int_0^t f(x_s)ds, \ t\geq 0, \end{aligned}$$

where $\gamma(\phi)=\phi(0)-g(\phi)$, in some detail and derived some stability theorems for the quasilinear system

$$(0.2.9)\qquad \begin{aligned} & x_{t_0} = \emptyset \\ & x(t) = \gamma(\emptyset) + g(x_t) + \int_{t_0}^{t} f(x_s)\,ds + \int_{t_0}^{t} F(s, x_s)\,ds \end{aligned}$$

Here F is a continuous mapping from $[\sigma, \infty) \times C_H$ into E^n, where $C_H = \{\emptyset \in C \mid \|\emptyset\| < H\}$, and further it is assumed to be Lipschitzian in the second argument on all of $[\sigma, \infty) \times C_H$ and $|F(t, \emptyset)| = o(\|\emptyset\|)$ uniformly in t as $\|\emptyset\| \to 0$

Hale and Cruz [20] formulate a class of hereditary systems which includes all functional differential equations of retarded type, Volterra integral equations and difference equations. This formulation also includes a class of FDENT, where the derivative $\dot{x}$ of the unknown function x occurs linearly in the equation in a very particular fashion. The class of equations considered by Driver [7] and Hale and Meyer [19] are particular cases of the above formulation. We give this formulation below

Let Ω denote the family of all compact subsets of the real line R and let $A \in \Omega$. Let zero be the maximal element of A Let $C(A) \equiv C(A, E^n)$ be the space of continuous functions with domain A and range in E^n with supremum norm. Let I be a connected interval of R, α be a continuous mapping of $I \times A$ into R such that $\alpha(t, A) \in \Omega$, $\alpha(t, \theta) \leq t$, $\alpha(t, \theta) \leq \alpha(t, \rho)$, $\alpha(t, 0) = t$, for all $t \in I$, $\theta \leq \rho \in A$. For any function x mapping the range of α into E^n, an operator $S: I x(x) \to C(A)$ is defined by the relation

$$(S_t x)(\theta) = x(\alpha(t, \theta)), \quad \theta \in A, \quad t \in I$$

The triplet (A,α,S) is called a hereditary structure. Let $g\ I\times C(A) \longrightarrow E^n$ and $f\ I\times C(A) \longrightarrow E^n$ A hereditary differential equation is a relation of the form

$$\frac{d}{dt}\,[D(t)S_t x] = f(t,S_t x), \tag{0.2 10}$$

where $D(t)\emptyset=\emptyset(0)-g(t,\emptyset)$, $t\in I$, $\emptyset\in C(A)$ Choosing g, A and α suitably, the equation (0 2.10) may be reduced to functional differential equations of retarded type, Volterra integral equations and difference equations If $A=[-\tau,0]$, $\tau>0$, $\alpha(t,\theta)=t+\theta$, then the equation (0,2,10) reduces to

$$\frac{d}{dt}\,[x(t)-g(t,x_t)] = f(t,x_t),$$

that is,

$$x(t) = g_t'(t,x_t) + g_\emptyset'(t,x_t)\,\dot{x}_t + f(t,x_t). \tag{0.2.11}$$

Clearly (0.2 11) is a neutral equation in which $\dot{x}_t$ occurs in a linear fashion.

In [20] , Hale and Cruz established theorems concerning existence, uniqueness and continuation of solutions of (0.2.10). They have also studied the continuous dependence of solutions of (0.2.10) on the initial data and parameters.

Here, as a remark, we mention that in this work we are studying a class of FDENT which neither includes those of Hale and Cruz [20] nor is included in their class of equations It is important to note that in this case the derivative $\dot{x}(t)$ of the unknown function x(t) need not to

occur linearly However, the norm, considered for our work, prevents us to consider some important neutral equations of the form (0.1.19) and (0.1.20).

Another aspect of neutral equations have been studied by Vasileva [38]. In [38], the behaviour of the solution of

$$(0.2.12) \qquad \dot{x}(t) = f(t, x(t), x(t-\tau), \dot{x}(t-\tau))$$

under the initial condition

$$(0.2.13) \qquad x(t) = \phi(t), \ t \in [0, \tau],$$

as $\tau \to 0$ has been investigated. Furthermore, asymptotic formulas with a remainder term of order τ^2, possessing a uniform character in the entire interval of variation of t, have been written for x and $\dot{x}$.

Rozhkov [33] established theorems concerning the existence of a periodic solution of period ω of the equation (0.2.12) for sufficiently small $\tau > 0$ with the assumption that the degenerate system

$$(0.2.14) \qquad \dot{x}(t) = f(t, x(t), x(t), \dot{x}(t)), x(0) = \phi(0)$$

has a periodic solution of period ω and an asymptotic representation in powers of the small lag τ for this solution has been constructed.

Brayton [3] has investigated the problem of existence of periodic solutions for small parameter ϵ of the equation

(0.2 15) $\quad x(t)-\alpha x(t-\tau)+\beta x(t)+\alpha\gamma x(t-\tau)=\varepsilon f(x(t))$

Hale and Cruz [21,22] in continuation of their study of FDENT (0 2 11), established the variation of constants formula and theorems on stability, uniform stability, uniform asymptotic stability of the trivial solution of (0.2 11). They also have discussed the boundedness of solutions of the nonhomogeneous systems

(0.2 16) $\quad \frac{d}{dt}[x(t) - g(t,x_t)] = f(t,x_t) + h(t)$

(0.2.17) $\quad \frac{d}{dt}[x(t) - g(t,x_t) - H(t)] = f(t,x_t),\ H(t_0) = 0$

where f,g,h are same as in (0 2.7) and $H \in C([\sigma,\infty), E^n)$.

0.3 CONTENTS OF THE PRESENT WORK

The present work has been divided into five chapters. We consider a class of neutral functional differential equations of the following type:

(0.3.1) $\quad \frac{dx}{dt} = f(t,x_t,\dot{x}_t)$,

where f $(a,b) \times D \longrightarrow E^n$, (a,b) is an interval in the real line R, E^n is Euclidean n-space and D is an open set in $AC([-\tau,0])$, the space of absolutely continuous functions $\psi\ [-\tau,0] \longrightarrow E^n$ Let AC((a,b)) denote the space of absolutely continuous functions $x:(a,b) \longrightarrow E^n$. For $x \in AC((a,b))$, the norm of x be given by

$$\|x\|_{(a,b)} = |x(a+)| + \int_a^b |\dot{x}(\theta)|\,d\theta$$

In equation (0 3 1), x_t stands for the translation of the restriction of $x \in AC((a-\tau,b))$ to $[t-\tau,t]$, $t \in (a,b)$, and is defined by $x_t(\theta) = x(t+\theta)$, $\theta \in [-\tau,0]$

By a solution $x(t,t_o,\emptyset)$ of (0 3.1) in $[t_o,t_o+\alpha]$, $\alpha>0$, $t_o+\alpha<b$, with initial function $\emptyset \in D$ at $t_o \in (a,b)$ we mean an absolutely continuous function $x(\ ,t_o,\emptyset)$ $[t_o-\tau,t_o+\alpha] \longrightarrow E^n$ such that $x_t \in D$, for $t \in [t_o,t_o+\alpha]$, $x_{t_o} = \emptyset$ and $x(t,t_o,\emptyset)$ satisfies the equation (0 3 1) for $t \in [t_o,t_o+\alpha]$

Chapter 1 deals with the local existence and uniqueness of solutions of the equation (0.3 1) when the right-hand side of this equation satisfies, besides other conditions, the Lipschitz condition. A theorem concerning the extended existence of solutions of (0.3.1) has been established. Continuous dependence of solutions of the equation (0.3 1) on initial data, parameter and right-hand side of the equation has been studied The basic tool for such study is the well-known Gronwall's lemma. Some examples have been considered to illustrate the applicability of the results

In Chapter 2 an effort has been made to relax the Lipschitz condition on f in (0 3 1) by some other regularity condition This enables us to establish simple existence of solutions of (0.3.1) under more general conditions Precisely sepaking, the Lipschitz condition is not implied by the regularity condition, which has been illustrated by an example, but the existence theorem in Chapter 2 (Theorem 12)

establishes, as a corollary, the existence of solutions of the autonomous neutral equation

$$(0.3.2) \qquad \frac{dx}{dt} = f(x_t, \dot{x}_t) ,$$

under the conditions of the corresponding theorem in Chapter 1 (Theorem 4). A theorem on the global existence of solutions of (0.3.1) has been proved.

Chapter 3 has been devoted to obtain a representation of solutions of (0.3.1) where $f(t,x_t,x_t)$ is linear and continuous in x for each t. Theorems on the existence of solutions of a nonhomogeneous linear system and of a system adjoint to the homogeneous linear system have been proved.

In Chapter 4 the equation (0.3.1) has been considered where $f(t,x_t,\dot{x}_t)$ is linear in x and periodic in t of period ω. Theorems concerning the existence of ω-periodic solutions of (0.3.1) and of a quasilinear system

$$(0.3.3) \qquad \frac{dx}{dt} = f(t,x_t,\dot{x}_t) + h(t,x_t,\dot{x}_t,\mu),$$

where $h\ (a,\infty) \times AC([-\tau,0]) \times [-\mu_0,\mu_0] \longrightarrow E^n$, $\mu_0 > 0$, have been established for sufficiently small μ.

The stability theory has been discussed in the last chapter. Two types of stability have been defined and theorems on stability, uniform stability, uniform asymptotic stability, exponential stability have been proved with the help of Liapunov functionals. Also some of the converse theorems have been obtained.

CHAPTER 1

EXISTENCE, UNIQUENESS AND CONTINUOUS DEPENDENCE OF SOLUTIONS WITH LIPSCHITZ CONDITION

1.1 NOTATIONS

The following notations are used throughout this work. Let E^n denote the Euclidean n-space and $|x|$ stand for any Euclidean norm. Let R and R^+ denote intervals $(-\infty,\infty)$ and $[0,\infty)$ respectively. For $t\in R$, $I_t=[t-\tau,t]$, $\tau>0$. Let $AC(I)$ denote the space of absolutely continuous functions $x: I \longrightarrow E^n$, where I is an interval of R, and for $x\in AC(I)$, the norm of x be given by

$$(1.1.1)\qquad \|x\|_I = |x(a+)| + \int_I |x(t)|\,dt,$$

where 'a' is the left end point of I. When $a\in I$ then $x(a+)$ is replaced by $x(a)$ in the definition of the norm of x. It is known that the space $AC(I)$ with the norm (1.1.1) is complete and hence is a Banach space. Let $BV(I)$ denote the space of functions $x: I \longrightarrow E^n$ of bounded variation. For $x\in BV(I)$,

$$(1.1.2)\qquad |x|_I = |x(a+)| + V(x,I),$$

where $V(x,I)$ stands for the total variation of x on I and is given by

$$V(x,I) = \sup \sum_{i=1}^{m} |x(b_i)-x(a_i)| ,$$

where the supremum is taken over all finite sets of points $a_i, b_i \in I$ with $a_1 \leq b_1 \leq a_2 \leq b_2 \leq \dots \leq a_m \leq b_m$. It is well-known that BV(I) with the norm (1 1 2) is a Banach space Some times we write $V(x,I)$ as $\underset{I}{V}x$. Let $L_1(I)$ denote the space of Lebesgue measurable functions x mapping I into E^n such that $|x(t)|$ is integrable in the sense of Lebesgue and, for $x \in L_1(I)$, the norm of x be given by

$$\| x \|_1 = \int_I |x(t)| dt$$

We shall denote by $L_\infty(I)$ the space of essentially bounded Lebesgue measurable functions x with domain I and range in E^n and for $x \in L_\infty(I)$,

$$\| x \|_\infty = \underset{t \in I}{\text{ess} \quad \sup} |x(t)|$$

For an lxm matrix A, we define $|A| = \sum_{i=1}^{l} \sum_{j=1}^{m} |a_{ij}|$ Let $AC_H(I_o) = \{ \psi \in AC(I_o) | \ \|\psi\|_{I_o} < H \}$, where H is any positive constant. For any interval (a,b) in R and for $x \in AC((a-\tau, b))$, we denote x_t as the translation of the restriction of x to the interval I_t, $t \in (a,b)$. In other words, for each $t \in (a,b)$, $x_t\ I_o \longrightarrow E^n$ defined by $x_t(\theta) = x(t+\theta)$ Hence $x_t \in AC(I_o)$ for each $x \in AC((a-\tau,b))$ and $t \in (a,b)$. It may be noted that, for $t \in (a,b)$, $\| x_t \|_{I_o} = \| x \|_{I_t}$ and $|x(t)| \leq \| x \|_{(a,b)}$. In this work integrals will be understood to be in the sense of Lebsegue or Lebesgue-Stieltjes.

1.2 EXISTENCE AND UNIQUENESS OF SOLUTIONS

In this chapter we establish a local existence and uniqueness theorem by the method of successive approximations and an extended existence theorem. Further continuous dependence of solutions on initial data, parameter and right-hand side of the equation has also been studied. The basic tool for this study is the well-known Gronwall's lemma Some examples have been considered as illustrations For the results in the present and following chapters we need the following

LEMMA 1 A set Q in $AC([a_1,b_1])$ is relatively compact if and only if

(i) it is uniformly bounded

(ii) $\lim\limits_{\sigma\to 0} \int_{a_1}^{b_1} |x(\theta+\sigma)- x(\theta)|\, d\theta = 0$ uniformly in $x\in Q$,

where for $\theta\in [a_1,b_1]$, we define $x(\theta+\sigma) = 0$ for $\theta+\sigma > b_1$ and $x(\theta+\sigma) = 0$ for $\theta+\sigma < a_1$.

Proof Let us assume that properties (i) and (ii) hold good To show that Q is relatively compact it is sufficient to prove that every sequence in Q has a subsequence which converges to a point in $AC([a_1,b_1])$.

Let Q^* be a family of functions $y\cdot(-\infty,\infty) \longrightarrow E^n$ defined by

$$y(t)=\begin{cases} x(a_1) & , \quad t\in(-\infty,a_1\,] \\ x(t) & , \quad t\in [a_1,b_1] \\ x(b_1) & , \quad t\in [b_1,\infty) \end{cases},$$

for $x \in Q$ Let $\Omega^* = \{ y | y \in Q^* \}$ So Ω^* $L_1(R)$

From assumption (i) it follows that Ω^* is uniformly bounded and assumption (ii) implies that

$$\lim_{\sigma \to 0} \int_{-\infty}^{\infty} |y(\theta+\sigma) - y(\theta)| \, d\theta = 0 \text{ uniformly in } y \in \Omega^*$$

Further it can be easily verified that $\lim_{\sigma^* \to \infty} \int_{|\theta| > \sigma^*} |y(\theta)| d\theta = 0$, uniformly in $y \in \Omega^*$ Hence from Frechet-Kolmogorov's theorem [41] it follows that Ω^* is relatively compact

Now we are in a position to prove that Q is relatively compact. Let $\langle x^n \rangle \subset Q$. Then $\langle y^n \rangle \subset Q^*$ and $\langle y^n \rangle \subset \Omega^*$ are sequences corresponding to $\langle x^n \rangle$ Since Ω^* is relatively compact, the sequence $\langle y^n \rangle$ has a subsequence $\langle y^{n_k} \rangle$, say which converges to a point y^* in $L_1(R)$. So we obtain

$$(1.2\ 1) \qquad \int_{a_1}^{b_1} |x^{n_k}(\theta) - y^*(\theta)| \, d\theta \to 0 \text{ as } k \to \infty$$

Corresponding to the subsequence $\langle y^{n_k} \rangle$ of $\langle y^n \rangle$ consider the subsequence $\langle x^{n_k} \rangle$ of $\langle x^n \rangle$ From assumption (i) it follows that $\langle x^{n_k}(a_1) \rangle \subset E^n$ is uniformly bounded. Consequently $\langle x^{n_k}(a_1) \rangle$ is relatively compact in E^n. Hence $\langle x^{n_k}(a_1) \rangle$ has a subsequence, say, $\langle x^{n_{k_i}}(a_1) \rangle$ such that

$$(1.2.2) \qquad |x^{n_{k_i}}(a_1) - \lambda_0| \to 0 \text{ as } i \to \infty,$$

where $\lambda_0 \in E^n$.

Define $x^*(t) = \lambda_0 + \int_{a_1}^{t} y^*(\theta) d\theta$, $t \in [a_1, b_1]$. So $x^* \in AC([a_1, b_1])$ with $x^*(a_1) = \lambda_0$ and $\dot{x}^*(t) = y^*(t)$ almost every where in $[a_1, b_1]$. From (1.2 1) and (1.2.2) it follows

that

$$\| x^{n_{k_i}} - x^*\|_{[a_1,b_1]} \to 0 \text{ as } i \to \infty.$$

This proves the sufficiency part of the lemma

In order to prove the necessity part of the lemma we suppose Q to be relatively compact. This immediately implies (i)

Suppose $\langle \dot{y}^n \rangle \subset \Omega^*$ So $\langle y^n \rangle \subset Q^*$ and $\langle x^n \rangle \subset Q$ Since Q is relatively compact, $\langle x^n \rangle$ has a subsequence $\langle x^{n_k} \rangle$, say, such that $\| x^{n_k} - x^{**}\|_{[a_1,b_1]} \to 0$ as $k \to \infty$, where $x^{**} \in AC([a_1,b_1])$. Defining y^{**} as

$$y^{**}(t) = \begin{cases} x^{**}(a_1), & t \in (-\infty, a_1] \\ x^{**}(t), & t \in [a_1, b_1] \\ x^{**}(b_1), & t \in [b_1, \infty), \end{cases}$$

we obtain

$$\int_{-\infty}^{\infty} |\dot{y}^{n_k}(\theta) - \dot{y}^{**}(\theta)| \, d\theta \to 0 \text{ as } k \to \infty$$

and this proves that Ω^* is relatively compact. Hence from Frechet-Kolmogorov's theorem we obtain

$$\int_{-\infty}^{\infty} |\dot{y}(\theta+\sigma) - \dot{y}(\theta)| \, d\theta \to 0 \text{ as } \sigma \to 0$$

uniformly in $\dot{y} \in \Omega^*$ From this (ii) follows.

This completes the proof of the lemma.

Let D be an open set in $AC(I_0)$ We consider a class of neutral functional differential equations of the following type.

(1 2 3) $\frac{dx}{dt} = f(t, x_t, \dot{x}_t)$,

where f $(a,b) \times D \longrightarrow E^n$.

<u>DEFINITION</u> 2 A function x is said to be a solution of (1.2.3) with initial function $\phi \in D$ at $t_0 \in (a,b)$ if there exists an $\alpha > 0$, $t_0 + \alpha \leq b$, such that $x\ [t_0 - \tau, t_0 + \alpha) \longrightarrow E^n$ is an absolutely continuous function with the properties

(i) $x_t \in D$ for $t \in [t_0, t_0 + \alpha)$

(ii) $x_{t_0} = \phi$

(iii) x satisfies (1 2.3) for $t \in [t_0, t_0 + \alpha)$

We shall denote this solution by $x(t, t_0, \phi)$

Let $t_0 \in (a,b)$ and $\phi \in D$. Since D is open, there exists an $r > 0$ such that $\phi \in N_\phi(r) \subset D$, where $N_\phi(r) = \{\psi \in AC(I_0) \mid \|\psi - \phi\|_{I_0} < r\}$. We define a function $\bar{\phi}$ $[t_0 - \tau, b) \longrightarrow E^n$ as follows

$$
(1\ 2.4) \qquad \begin{aligned} \bar{\phi}_{t_0}(\theta) &= \phi(\theta), \ \theta \in I_0 \\ \bar{\phi}(t) &= \phi(0), \ t \in [t_0, b). \end{aligned}
$$

In what follows we show that there exists a β_1, $0 < \beta_1 \leq \tau$, such that $\bar{\phi}_t \in D$ for $t \in [t_0, t_0 + \beta_1]$, $t_0 + \beta_1 < b$. For this it is sufficient to show that $\|\bar{\phi}_t - \phi\|_{I_0} < r$, $t \in [t_0, t_0 + \beta_1]$ Now, for $t \in [t_0, t_0 + \beta_1]$,

$$
\|\bar{\phi}_t - \phi\|_{I_0} = |\bar{\phi}(t-\tau) - \phi(-\tau)| + \int_{-\tau}^{0} |\dot{\bar{\phi}}(t+\theta) - \dot{\phi}(\theta)| \, d\theta
$$

$$
= |\phi(t-\tau-t_0) - \phi(-\tau)| + \int_{t-\tau}^{t} | \dot{\bar{\phi}}(\theta) - \dot{\phi}(\theta - t)| \, d\theta
$$

$$= |\phi(t-\tau-t_0)-\phi(-\tau)| + \int_{t-\tau}^{t_0} |\bar{\phi}(\theta)-\phi(\theta-t)|\, d\theta$$

$$+ \int_{t_0}^{t} |\phi(\theta-t)|\, d\theta$$

$$= |\phi(t-\tau-t_0)-\phi(-\tau)| + \int_{t-\tau}^{t_0} |\phi(\theta-t_0)-\phi(\theta-t)|\, d\theta$$

$$+ \int_{t_0}^{t} |\phi(\theta-t)|\, d\theta$$

$$= |\phi(t-\tau-t_0)-\phi(-\tau)| + \int_{-\tau}^{t_0-t} |\phi(u+t-t_0)-\phi(u)|\, du$$

$$+ \int_{t_0-t}^{0} |\phi(u)|\, du$$

$$\leq |\phi(t-\tau-t_0)-\phi(-\tau)| + \int_{-\tau}^{0} |\phi(u+t-t_0)-\phi(u)|\, du$$

$$+ \int_{t_0-t}^{0} |\phi(u)|\, du \,.$$

Since ϕ is continuous, ϕ is integrable and $\{\phi\}$, singleton ϕ, is relatively compact, then from Lemma 1 it follows that given $r>0$ we can select δ_1, δ_2 and δ_3 such that for $t-t_0 \leq \beta_1$, where $\beta_1 = \min(\delta_1, \delta_2, \delta_3, \tau)$, we obtain $|\phi(t-\tau-t_0)-\phi(-\tau)| < r/6$, $\int_{-\tau}^{0} |\phi(u+t-t_0) - \phi(u)|\, du < r/6$ and $\int_{t_0-t}^{0} |\phi(u)|\, du < r/6$.

Hence $\|\bar{\phi}_t-\phi\|_{I_0} < r/2$ for $t \in [t_0, t_0+\beta_1]$

Suppose $0<r^*<r/4$ and $N_{\bar{\phi}}(r^*) = \{x \in AC([t_0-\tau, t_0+\beta_1]) \mid \|x-\bar{\phi}\|_{[t_0-\tau, t_0+\beta_1]} < r^*\}$. Then we claim that $x_t \in D$ if $x \in \bar{N}_{\bar{\phi}}(r^*)$ for each $t \in [t_0, t_0+\beta_1]$. Here $\bar{N}_{\bar{\phi}}(r^*)$ denotes the closure of

$N_{\bar{\phi}}(r^*)$. For this, as in the previous case, it is to be shown that $x_t \in N_{\phi}(r)$ when $x \in \bar{N}_{\bar{\phi}}(r^*)$ and $t \in [t_o, t_o+\beta_1]$. Now, for such an x and t, we have

$$\| x_t - \phi \|_{I_o} \leq \| x_t - \bar{\phi}_t \|_{I_o} + \| \bar{\phi}_t - \phi \|_{I_o}$$

$$\leq |x(t-\tau) - \bar{\phi}(t-\tau)| + \int_{t-\tau}^{t} |x(\theta) - \bar{\phi}(\theta)| d\theta + \| \bar{\phi}_t - \phi \|_{I_o}$$

$$\leq 2 \| x - \bar{\phi} \|_{[t_o-\tau, t_o+\beta_1]} + \| \bar{\phi}_t - \phi \|_{I_o}$$

$$< 2r^* + r/2$$

$$< r .$$

This proves our assertion.

<u>DEFINITION</u> 3 For $a < \eta_1 < \eta_2 < b$, a function $x \; [\eta_1 - \tau, \eta_2] \longrightarrow E^n$ is said to be admissible if

(i) it is absolutely continuous,

(ii) $x_t \in D$ for $t \in [\eta_1, \eta_2]$.

It may be noted that η_1 and η_2 may be different for different functions

Now we prove a local existence and uniqueness theorem for the equation (1.2 3)

<u>THEOREM</u> 4 Suppose that the following conditions are satisfied

(C_1) $f(t, x_t, \dot{x}_t)$ is measurable in t for each fixed admissible x,

(C_2) $|f(t, x_t, \dot{x}_t)| \leq m(t)$ for all admissible x, where $m \; (a,b) \longrightarrow R^+$ is integrable,

(C_3) $|f(t, \psi, \dot{\psi}) - f(t, \psi^*, \dot{\psi}^*)| \leq L \| \psi - \psi^* \|_{I_o}$ for $t \in (a,b)$ and $\psi, \psi^* \in D$, where L is a positive constant.

Then, for $t_o \in (a,b)$ and $\phi \in D$, there exists an α, $0 < \alpha < \tau$, such that the system (1.2.3) admits a unique solution $x(t,t_o,\phi)$ in $[t_o, t_o+\alpha]$ with initial function ϕ at t_o.

Proof: We define a function $M: [t_o-\tau, b) \longrightarrow R^+$ by

$$(1.2.5) \qquad M(t) = \begin{cases} 0 & , \ t \in I_{t_o} \\ \int_{t_o}^{t} m(s)ds, & t \in [t_o, b) \ . \end{cases}$$

So $M(t)$ is absolutely continuous and monotonic increasing with t.

Further, we define a function $\bar{\phi}: [t_o-\tau, b) \longrightarrow E^n$ as in (1.2.4). Since D is open, there exists an $r>0$ such that $\phi \in N_{\phi}(r) \subset D$. From the above discussion it follows that there exists a β_1, $0<\beta_1 \leq \tau$, $t_o+\beta_1<b$, such that $\bar{\phi}_t \in D$ for $t \in [t_o, t_o+\beta_1]$ and $x_t \in D$ for $x \in \bar{N}_{\bar{\phi}}(r^*)$ and $t \in [t_o, t_o+\beta_1]$, where $0<r^*<r/4$.

We choose an α such that $0<\alpha<\beta_1$ and $M(t_o+\alpha) < r^*$. Such a choice is always possible because $M(t_o) = 0$ and $M(t)$ is continuous.

We define

$$(1.2.6) \qquad \begin{aligned} x^o(t) &= \bar{\phi}(t) \ , \ t \in [t_o-\tau, t_o+\alpha] \ , \text{ and} \\ x^k_{t_o}(\theta) &= \phi(\theta) \ , \ \theta \in I_o \\ x^k(t) &= \phi(0) + \int_{t_o}^{t} f(s, x^{k-1}_s, \dot{x}^{k-1}_s)ds, \ t \in [t_o, t_o+\alpha] \ , \end{aligned}$$

where $k = 1,2....$

Now we prove that x^ks are well-defined. Clearly x^1 is defined. Suppose that $x^2, x^3, ..., x^k$ are well-defined. To show that x^{k+1} is also defined, it is sufficient to prove that x^k

is a restriction to $[t_o-\tau,t_o+\alpha]$ of a function in $N_{\bar{\emptyset}}(r^*)$. We define a function $\bar{x}^k:[t_o-\tau,t_o+\beta_1]\longrightarrow E^n$ as follows:

$$(1.2.7)\qquad \bar{x}^k(t)=\begin{cases} x^k(t) & ,\ t\in[t_o-\tau,t_o+\alpha] \\ x^k(t_o+\alpha) & ,\ t\in[t_o+\alpha,t_o+\beta_1]\ . \end{cases}$$

Clearly $\bar{x}^k$ is an absolutely continuous function. Now

$$\begin{aligned} \|\bar{x}^k-\bar{\emptyset}\|_{[t_o-\tau,t_o+\beta_1]} &= |\bar{x}^k(t_o-\tau)-\bar{\emptyset}(t_o-\tau)|+\int_{t_o-\tau}^{t_o+\beta_1}|\dot{\bar{x}}^k(t)-\dot{\bar{\emptyset}}(t)|\,dt \\ &= \int_{t_o-\tau}^{t_o+\alpha}|\dot{x}^k(t)-\dot{\bar{\emptyset}}(t)|\ dt \\ &= \int_{t_o}^{t_o+\alpha}|f(t,x_t^{k-1},\dot{x}_t^{k-1})|\ dt \\ &\leq \int_{t_o}^{t_o+\alpha} m(t)\ dt \\ &\leq M(t_o+\alpha) \\ &< r^*\ . \end{aligned}$$

This proves that $\bar{x}^k\in N_{\bar{\emptyset}}(r^*)$. Hence from the definition of $\bar{x}^k$ our assertion follows. By induction it follows that x^ks are well-defined for all $k=1,2,\ldots$.

In the following we show that the sequence $\langle x^k\rangle$ is Cauchy in $AC([t_o-\tau,t_o+\alpha])$. For this we first prove that

$$\|x^{k+1}-x^k\|_{[t_o-\tau,t_o+\alpha]}\leq M(t_o+\alpha)\ \frac{L_\alpha^k k}{k!}$$

whence from the triangle inequality the result is obvious. Now

$$(1.2.8) \quad \| x^{k+1} - x^k \|_{[t_0-\tau, t_0+\alpha]}$$

$$= |x^{k+1}(t_0-\tau) - x^k(t_0-\tau)| + \int_{t_0-\tau}^{t_0+\alpha} |\dot{x}^{k+1}(t) - \dot{x}^k(t)| \, dt$$

$$= \int_{t_0}^{t_0+\alpha} | \dot{x}^{k+1}(t) - \dot{x}^k(t)| \, dt$$

$$= \int_{t_0}^{t_0+\alpha} | f(t, x_t^k, \dot{x}_t^k) - f(t, x_t^{k-1}, \dot{x}_t^{k-1})| \, dt$$

$$\leq L \int_{t_0}^{t_0+\alpha} \| x_t^k - x_t^{k-1} \|_{I_0} \, dt$$

Further, for $t \in [t_0, t_0+\alpha]$, we obtain

$$(1.2.9) \quad \| x^k - x^{k-1} \|_{I_t}$$

$$= |x^k(t-\tau) - x^{k-1}(t-\tau)| + \int_{t-\tau}^{t} |\dot{x}^k(\theta) - \dot{x}^{k-1}(\theta)| \, d\theta$$

$$= \int_{t_0}^{t} | \dot{x}^k(\theta) - \dot{x}^{k-1}(\theta)| \, d\theta$$

$$= \int_{t_0}^{t} | f(\theta, x_\theta^{k-1}, \dot{x}_\theta^{k-1}) - f(\theta, x_\theta^{k-2}, \dot{x}_\theta^{k-2})| \, d\theta$$

$$\leq L \int_{t_0}^{t} \| x_\theta^{k-1} - x_\theta^{k-2} \|_{I_0} \, d\theta$$

$$\leq L \int_{t_0}^{t} \| x^{k-1} - x^{k-2} \|_{I_\theta} \, d\theta \quad .$$

But, for $t \in [t_0, t_0+\alpha]$,

$$(1.2.10) \quad \| x^1-x^o\|_{I_t}$$

$$= |x^1(t-\tau)-x^o(t-\tau)|+ \int_{t-\tau}^{t} |\dot{x}^1(\theta)-\dot{x}^o(\theta)| \; d\theta$$

$$\leq \int_{t_o}^{t_o+\alpha} | \dot{x}^1(\theta)| \; d\theta$$

$$\leq \int_{t_o}^{t_o+\alpha} | f(\theta,x^o_\theta,\dot{x}^o_\theta)| \; d\theta$$

$$\leq M(t_o+\alpha).$$

From (1.2.9) and (1.2.10) it follows that

$$(1.2.11) \quad \|x^k-x^{k-1}\|_{I_t} \leq L^{k-1} M(t_o+\alpha) \frac{(t-t_o)^{k-1}}{(k-1)!}$$

Using (1.2.11) in (1.2.8) we obtain

$$(1.2.12) \quad \| x^{k+1}- x^k\|_{[t_o-\tau, t_o+\alpha]} \leq \frac{M(t_o+\alpha)L^k\alpha^k}{k!}$$

From (1.2.12), our assertion that the sequence $\langle x^k \rangle$ is Cauchy in $AC([t_o-\tau,t_o+\alpha])$ follows because

$$\| x^m-x^n\|_{[t_o-\tau,t_o+\alpha]} \leq \sum_{k=n}^{m-1} \| x^{k+1}-x^k\|_{[t_o-\tau,t_o+\alpha]}$$

$$\leq \sum_{k=n}^{m-1} M(t_o+\alpha) \frac{L^k\alpha^k}{k!}$$

and $\sum_{k=n}^{m-1} \frac{L^k\alpha^k}{k!} \to 0$ as $n \to \infty$ and $m \to \infty$.

Since the space $AC([t_o-\tau,t_o+\alpha])$ with the norm $\|\cdot\|_{[t_o-\tau,t_o+\alpha]}$ is complete, there exists an element x^o in $AC([t_o-\tau,t_o+\alpha])$ such that

(1.2.13) $\qquad \|x^k - x^o\|_{[t_o-\tau,t_o+\alpha]} \to 0$ as $k \to \infty$.

Now we prove that x^o is an admissible function. We define functions $\bar{x}^k$ as in (1.2.7) and a function $\bar{x}^o : [t_o-\tau, t_o+\beta_1] \longrightarrow E^n$ as follows:

$$(1.2.14) \qquad \bar{x}^o(t) = \begin{cases} x^o(t) & , \ t \in [t_o-\tau, t_o+\alpha] \\ x^o(t_o+\alpha), & t \in [t_o+\alpha, t_o+\beta_1] \end{cases} .$$

So $\bar{x}^k$, $\bar{x}^o \in AC([t_o-\tau, t_o+\beta_1])$. Further, we have shown that $\bar{x}^k \in N_{\bar{\emptyset}}(r^*)$ and since

$$\|\bar{x}^k - \bar{x}^o\|_{[t_o-\tau,t_o+\beta_1]} = \|x^k - x^o\|_{[t_o-\tau,t_o+\alpha]},$$

from (1.2.13) it follows that $\bar{x}^o \in N_{\bar{\emptyset}}(r^*)$. Hence $\bar{x}^o_t \in D$ and so x^o is an admissible function.

$$\text{Since } \|x^o_{t_o} - \emptyset\|_{I_o} = \|x^o_{t_o} - x^k_{t_o}\|_{I_o} \leq \|x^o - x^k\|_{[t_o-\tau,t_o+\alpha]},$$

from (1.2.13) we obtain $x^o_{t_o} = \emptyset$.

In the following we show that $x^o(t)$, $t \in [t_o, t_o+\alpha]$, satisfies the integral equation

$$(1.2.15) \qquad x(t) = \emptyset(0) + \int_{t_o}^{t} f(s, x_s, \dot{x}_s)\, ds .$$

Since, for $t \in [t_o, t_o+\alpha]$,

$$|x^k(t) - x^o(t)| \leq \|x^k - x^o\|_{[t_o-\tau,t_o+\alpha]},$$

from (1.2.13) we obtain

$$\lim_{k\to\infty} x^k(t) = x^o(t).$$

Also since, for $t \in [t_o, t_o+\alpha]$,

$$\| x^k - x^o \|_{I_t} = \int_{t-\tau}^{t} | \dot{x}^k(\theta) - \dot{x}^o(\theta) | \, d\theta$$

$$\leq \int_{t_o-\tau}^{t_o+\alpha} |\dot{x}^k(\theta) - \dot{x}^o(\theta)| \, d\theta$$

$$\leq \| x^k - x^o \|_{[t_o-\tau, t_o+\alpha]} ,$$

we have

$$\left| \int_{t_o}^{t} f(s, x_s^{k-1}, \dot{x}_s^{k-1}) \, ds - \int_{t_o}^{t} f(s, x_s^o, \dot{x}_s^o) ds \right|$$

$$\leq \int_{t_o}^{t} | f(s, x_s^{k-1}, \dot{x}_s^{k-1}) - f(s, x_s^o, \dot{x}_s^o) | \, ds$$

$$\leq L \int_{t_o}^{t} \| x_s^{k-1} - x_s^o \|_{I_o} \, ds$$

$$\leq L \| x^{k-1} - x^o \|_{[t_o-\tau, t_o+\alpha]} \int_{t_o}^{t} ds$$

$$\leq L \alpha \| x^{k-1} - x^o \|_{[t_o-\tau, t_o+\alpha]} .$$

Hence passing to the limit as $k \to \infty$ in (1.2.6) we obtain

$$x^o(t) = \phi(0) + \int_{t_o}^{t} f(s, x_s^o, \dot{x}_s^o) \, ds$$

for $t \in [t_o, t_o+\alpha]$. This proves that $x^o(t)$ is a solution of (1.2.3) with initial function ϕ at t_o.

In the following we show that the solution $x^o(t, t_o, \phi)$ is unique. Let $y^o(t, t_o, \phi)$ be another solution of (1.2.3) in $[t_o, t_o+\alpha]$ with initial function ϕ at t_o. We set $x^o(t) = x^o(t, t_o, \phi)$ and $y^o(t) = y^o(t, t_o, \phi)$. Now we prove that $x^o(t) = y^o(t)$ for $t \in [t_o, t_o+\alpha]$. If possible let there exist a $t_1 \in (t_o, t_o+\alpha]$ such that $x^o(t_1) \neq y^o(t_1)$. Let

$|x^o(t_1) - y^o(t_1)| = c_o > 0$. Since $|x^o(t_1) - y^o(t_1)| \leq$ $\|x^o-y^o\|_{[t_o-\tau,t_o+\alpha]}$, then it is possible to set $\|x^o-y^o\|_{[t_o-\tau,t_o+\alpha]} = c_1$ where $c_1 > 0$. But

$$(1.2.16) \qquad \|x^o-y^o\|_{[t_o-\tau,t_o+\alpha]} = \int_{t_o}^{t_o+\alpha} |\dot{x}^o(t) - \dot{y}^o(t)|\, dt$$

$$= \int_{t_o}^{t_o+\alpha} |f(t,x_t^o,\dot{x}_t^o) - f(t,y_t^o,\dot{y}_t^o)|\, dt$$

$$\leq L \int_{t_o}^{t_o+\alpha} \|x_t^o - y_t^o\|_{I_o}\, dt$$

and, for $t \in [t_o, t_o+\alpha]$,

$$(1.2.17) \qquad \|x^o-y^o\|_{I_t} = \int_{t_o}^{t} |\dot{x}^o(s) - \dot{y}^o(s)|\, ds$$

$$\leq L \int_{t_o}^{t} \|x_s^o - y_s^o\|_{I_o}\, ds .$$

Again since, for $t \in [t_o, t_o+\alpha]$,

$$\|x^o-y^o\|_{I_t} \leq \|x^o-y^o\|_{[t_o-\tau,t_o+\alpha]}$$

$$\leq c_1 ,$$

from (1.2.17) it follows that

$$\|x^o-y^o\|_{I_t} \leq L\, c_1(t-t_o)$$

for $t \in [t_o, t_o+\alpha]$. Repeating the process n-times we obtain

$$(1.2.18) \qquad \|x^o-y^o\|_{I_t} \leq L^n c_1 \frac{(t-t_o)^n}{n!}$$

$$\leq L^n c_1 \frac{\alpha^n}{n!} .$$

From (1.2.16) and (1.2.18) we obtain

$$\| x^o - y^o \|_{[t_o-\tau, t_o+\alpha]} \leq L^{n+1} c_1 \frac{\alpha^{n+1}}{n!} \quad .$$

So $\| x^o - y^o \|_{[t_o-\tau, t_o+\alpha]} \to 0$ as $n \to \infty$, that is, $\| x^o - y^o \|_{[t_o-\tau, t_o+\alpha]} = 0$.

Consequently, $|x^o(t_1) - y^o(t_1)| = 0$. This contradicts our assumption that $|x^o(t_1) - y^o(t_1)| = c_o > 0$. So $x^o \equiv y^o$. This completes the proof of the theorem.

In the following we prove an extended existence theorem for the system (1.2.3).

<u>THEOREM</u> 5. Suppose that the conditions of Theorem 4 are satisfied. Then, for $t_o \in (a,b)$ and $\phi \in D$, there exists a $\beta, t_o < \beta \leq b$, such that the system (1.2.3) admits a unique solution $x(t, t_o, \phi)$ in $[t_o, \beta)$ with initial function ϕ at t_o. If $\beta < b$ and β can not be increased then $\bar{x}_\beta$ is not an interior point of D, where $\bar{x}: [t_o - \tau, b) \longrightarrow E^n$ is defined as follows:

$$\bar{x}(t) = \begin{cases} x(t, t_o, \phi) & , t \in [t_o - \tau, \beta) \\ x(\beta - 0, t_o, \phi) & , t \in [\beta, b) \end{cases} \quad .$$

The existence of $x(\beta - 0, t_o, \phi)$ has been shown below.

<u>Proof</u>: For $t_o \leq t_1 < t_2 < \beta$, we obtain

$$(1.2.19) \qquad |x(t_2, t_o, \phi) - x(t_1, t_o, \phi)| \leq \left| \int_{t_1}^{t_2} |f(s, x_s, \dot{x}_s)| ds \right|$$
$$\leq |M(t_2) - M(t_1)|$$

Since $M(t)$ is uniformly continuous on $[t_o,t_2]$, then $|t_2-t_1|\to 0$ implies that $|M(t_2)-M(t_1)|\to 0$. Consequently, from (1.2.19) we have

$$(1.2.20)\qquad |x(t_2,t_o,\phi) - x(t_1,t_o,\phi)| \to 0$$

as $|t_2-t_1|\to 0$. Let $< t_m >\subset[t_o,\beta)$ with $t_m\to\beta-0$ as $m\to\infty$. Consider the sequence $< x(t_m,t_o,\phi) >$ in E^n. Since $<t_m>$ is Cauchy, then from (1.2.20) it follows that

$$|x(t_m,t_o,\phi) - (t_k,t_o,\phi)| \longrightarrow 0$$

as $m\to\infty$ and $k\to\infty$. This proves that the sequence $<x(t_m,t_o,\phi)>$ is Cauchy. From the completeness of E^n it follows that $\lim_{m\to\infty} x(t_m,t_o,\phi)$ exists, that is, $\lim_{t\to\beta-0} x(t,t_o,\phi)$ exists. Setting $\lim_{t\to\beta-0} x(t,t_o,\phi) = x(\beta-0,t_o,\phi)$ we see that $x(\beta-0,t_o,\phi)$ exists.

On the contrary, let us assume that $\bar{x}_\beta$ is an interior point of D. Set $\tilde{\phi} = \bar{x}_\beta$. By Theorem 4 there exists a unique solution $x^*(t,\beta,\tilde{\phi})$ of (1.2.3) in $[\beta,\beta+\alpha]$, $0<\alpha<\tau$, with initial function $\tilde{\phi}$ at β.

We define a function $\tilde{x}:[t_o-\tau,\beta]\longrightarrow E^n$ as follows:

$$\tilde{x}(t) = \begin{cases} x(t,t_o,\phi) & , t\in[t_o-\tau,\beta) \\ x(\beta-0,t_o,\phi) & , t=\beta\ . \end{cases}$$

Clearly $\tilde{x}$ is an absolutely continuous function with $\tilde{x}_{t_o} = \phi$. Since $x_t(t_o,\phi)\in D$ for $t\in[t_o,\beta)$, then $\tilde{x}_t\in D$ for $t\in[t_o,\beta)$. Further, since $\tilde{x}_\beta=\bar{x}_\beta$, from our assumption it follows that $\tilde{x}_\beta\in D$. Also $\tilde{x}(t)$ satisfies (1.2.3) for $t\in[t_o,\beta]$. Hence $\tilde{x}(t)$ is a solution

of the system (1.2.3) in $[t_0,\beta]$ with initial function ϕ at t_0.

Next we define a function $\hat{x}: [t_0-\tau,\beta+\alpha] \longrightarrow E^n$ as follows:

$$\hat{x}(t) = \begin{cases} \tilde{x}(t,t_0,\phi) & , t \in [t_0-\tau,\beta] \\ x^*(t,\beta,\tilde{\phi}) & , t \in [\beta,\beta+\alpha] \end{cases} .$$

So $\hat{x}$ is an absolutely continuous function with $\hat{x}_{t_0} = \phi$ and $\hat{x}_t \in D$ for $t \in [t_0,\beta+\alpha]$. To show that $\hat{x}$ is a solution of (1.2.3) in $[t_0, \beta+\alpha]$ with initial function ϕ at t_0, it is enough to prove that $\hat{x}$ satisfies the integral equation

$$(1.2.21) \qquad x(t) = \phi(0) + \int_{t_0}^{t} f(s,x_s,\dot{x}_s)ds$$

for $t \in [t_0,\beta+\alpha]$. Since $\tilde{x}(t)$ is a solution of (1.2.3) in $[t_0,\beta]$, then $\hat{x}(t)$ satisfies (1.2.21) for $t \in [t_0,\beta]$. Further we have

$$x^*(t) = \tilde{\phi}(0) + \int_{\beta}^{t} f(s,x_s^*,\dot{x}_s^*)\, ds ,$$

for $t \in [\beta,\beta+\alpha]$, that is,

$$(1.2.22) \qquad \hat{x}(t) = \tilde{\phi}(0) + \int_{\beta}^{t} f(s,\hat{x}_s,\dot{\hat{x}}_s)\, ds,$$

for $t \in [\beta,\beta+\alpha]$. But

$$\tilde{\phi}(0) = \phi(0) + \int_{t_0}^{\beta} f(s,\tilde{x}_s,\dot{\tilde{x}}_s)\, ds$$

$$= \phi(0) + \int_{t_0}^{\beta} f(s,\hat{x}_s,\dot{\hat{x}}_s)\, ds .$$

Hence from (1.2.22) it follows that, for $t \in [\beta,\beta+\alpha]$,

$$\hat{x}(t) = \phi(0) + \int_{t_0}^{t} f(s,\hat{x}_s,\dot{\hat{x}}_s)\, ds.$$

Consequently $\hat{x}(t)$ satisfies (1.2.21) for $t\in[t_o,\beta+\alpha]$. From this it follows that β can be extended. This contradiction completes the proof of the theorem.

1.3 CONTINUOUS DEPENDENCE OF SOLUTIONS.

In addition to the system (1.2.3), we consider the following systems:

(1.3.1) $$\frac{dy}{dt} = f(t,y_t,\dot{y}_t) + H_1(t,y_t,\dot{y}_t),$$

where H_1: $(a,b) \times D \longrightarrow E^n$, and

(1.3.2) $$\frac{dy}{dt} = f(t,y_t,\dot{y}_t,\lambda) \ ,$$

where f: $(a,b) \times D \times E^k \longrightarrow E^n$.

THEOREM 6. Consider the system (1.3.1) where $f(t,y_t,\dot{y}_t)$ satisfies the assumptions of Theorem 4 and $H_1(t,y_t,\dot{y}_t)$ is integrable as a function of t for each fixed admissible y and is such that, for $t_o\in(a,b)$ and $\phi^*\in D$, there exists an α^*, $0<\alpha^*<\tau$, such that the system (1.3.1) admits a unique solution $y(t,t_o,\phi^*)$ in $[t_o,t_o+\alpha^*]$ with initial function ϕ^* at t_o.

Let $x(t,t_o,\phi)$, for $\phi\in D$, be a unique solution of (1.2.3) in $[t_o,t_o+\alpha]$, $0<\alpha<\tau$, with initial function ϕ at t_o. Let $\beta=\min\{t_o+\alpha,t_o+\alpha^*\}$. Then for $\epsilon>0$ there exists a $\delta=\delta(\epsilon,t_o,\phi,H_1) > 0$ such that $||\phi-\phi^*||_{I_o} < \delta$ and $|H_1(t,y_t,\dot{y}_t)|<\delta$ imply

$$|y(t,t_o,\phi^*) - x(t,t_o,\phi)| < \epsilon \text{ for } t\in[t_o,\beta] .$$

Proof: We set $x(t)=x(t,t_o,\phi)$ and $y(t)=y(t,t_o,\phi^*)$. For $t\in[t_o,\beta]$

$$(1.3.3) \qquad |x(t)-y(t)| \leq |\emptyset(0)-\emptyset^*(0)| + \int_{t_o}^{t} |f(s,x_s,\dot{x}_s) - f(s,y_s,\dot{y}_s)|ds + \int_{t_o}^{t} |H_1(s,y_s,\dot{y}_s)|ds$$

$$\leq \|\emptyset-\emptyset^*\|_{I_o} + L\int_{t_o}^{t} \|x_s-y_s\|_{I_o}\, ds + \int_{t_o}^{t} |H_1(s,y_s,\dot{y}_s)|ds$$

Further, for $t\in[t_o,\beta]$, we obtain

$$(1.3.4) \qquad \|x-y\|_{I_t} = |x(t-\tau)-y(t-\tau)| + \int_{t-\tau}^{t} |\dot{x}(s)-\dot{y}(s)|ds$$

$$\leq |\emptyset(t-\tau-t_o)-\emptyset^*(t-\tau-t_o)| + \int_{t-\tau}^{t_o} |\dot{\emptyset}(s-t_o)-\dot{\emptyset}^*(s-t_o)|ds + \int_{t_o}^{t} |f(s,x_s,\dot{x}_s)-f(s,y_s,\dot{y}_s)|ds + \int_{t_o}^{t} |H_1(s,y_s,\dot{y}_s)|\, ds$$

$$\leq 2\|\emptyset-\emptyset^*\|_{I_o} + L\int_{t_o}^{t} \|x_s-y_s\|_{I_o} ds + \int_{t_o}^{\beta} |H_1(s,y_s,\dot{y}_s)|ds$$

Applying Gronwall's lemma to (1.3.4) we get

$$(1.3.5) \qquad \|x-y\|_{I_t} \leq K_1\, e^{L(t-t_o)},$$

for $t\in[t_o,\beta]$, where $K_1=2\|\emptyset-\emptyset^*\|_{I_o} + \int_{t_o}^{\beta} |H_1(s,y_s,\dot{y}_s)|ds$.

Now from (1.3.3) and (1.3.5) it follows that

$$|x(t)-y(t)| \leq \| \phi-\phi^* \|_{I_0} + K_1 e^{L(t-t_0)} + \int_{t_0}^{\beta} |H_1(s,y_s,\dot{y}_s)|\, ds$$

$$\leq (1+2\, e^{L(\beta-t_0)}) \| \phi-\phi^* \|_{I_0}$$

$$+(1+e^{L(\beta-t_0)}) \int_{t_0}^{\beta} |H_1(s,y_s,\dot{y}_s)|\, ds$$

$$\leq (1+2\, e^{L(\beta-t_0)})\, \delta + (1+e^{L(\beta-t_0)})(\beta-t_0)\delta \, .$$

Choosing δ suitably, it is easy to see that $|x(t)-y(t)| < \varepsilon$ for $t \in [t_0, \beta]$. This completes the proof of the theorem.

THEOREM 7. Suppose that f in the equation (1.2.3) satisfies conditions of Theorem 4. Let $x(t,t_0,\phi)$ and $y(t,t_0,\phi^*)$ be any two solutions of the system (1.2.3) in $[t_0, t_0+\alpha]$ and $[t_0, t_0+\alpha^*]$, $0<\alpha, \alpha^* < \tau$, with initial functions ϕ and ϕ^* at t_0 respectively, where $\phi, \phi^* \in D$ and $t_0 \in (a,b)$. Then for every $\varepsilon>0$ there exists a $\delta=\delta(\varepsilon, t_0, \phi)>0$ such that $\| \phi-\phi^* \|_{I_0} < \delta$ implies that

$$|x(t,t_0,\phi)-y(t,t_0,\phi^*)| < \varepsilon \text{ for } t \in [t_0 \min\{t_0+\alpha, t_0+\alpha^*\}].$$

The proof of this theorem follows from that of Theorem 6 when H_1 in the system (1.3.1) is identically zero.

THEOREM 8. Consider equation (1.3.2) where, for each $\lambda \in E^k$, f is assumed to satisfy conditions (C_1) and (C_2) of Theorem 4. Further f satisfies a Lipschitz condition of the following type:

$$|f(t,\psi,\dot{\psi},\lambda)-f(t,\psi^*,\dot{\psi}^*,\lambda^*)| \leq L^* [\| \psi-\psi^* \|_{I_0} + |\lambda-\lambda^*|] ,$$

for all $t\in(a,b)$, $\psi,\psi^*\in D$ and $\lambda,\lambda^*\in E^k$. For each $\lambda\in E^k$, the existence of a unique solution of (1.3.2) is assured by Theorem 4. Let $x(t,t_0,\emptyset\,\lambda)$ and $y(t,t_0,\emptyset^*,\lambda^*)$ be any two solutions of (1.3.2) corresponding to parameters λ and λ^* respectively in $[t_0,t_0+\alpha]$ and $[t_0,t_0+\alpha^*]$, $0<\alpha,\alpha^*<\tau$, with initial functions $\emptyset$ and $\emptyset^*$ at t_0, where $t_0\in(a,b)$ and $\emptyset,\emptyset^*\in D$. Let $\beta=\min\{t_0+\alpha,\ t_0+\alpha^*\}$. Then for every $\varepsilon>0$ there exists a $\delta=\delta(\varepsilon,t_0,\emptyset,\lambda)>0$ such that $\|\emptyset-\emptyset^*\|_{I_0}<\delta$ and $|\lambda-\lambda^*|<\delta$ imply that

$$|x(t,t_0,\emptyset,\lambda)-y(t,t_0,\emptyset^*,\lambda^*)|<\varepsilon \text{ for } t\in[t_0,\beta]\,.$$

<u>Proof:</u> We set $x(t)=x(t,t_0,\emptyset,\lambda)$ and $y(t)=y(t,t_0,\emptyset^*,\lambda^*)$. Now, for $t\in[t_0,\beta]$,

$$(1.3.6)\qquad |x(t)-y(t)|\le|\emptyset(0)-\emptyset^*(0)| + \int_{t_0}^{t}|f(s,x_s,\dot{x}_s,\lambda)-f(s,y_s,\dot{y}_s,\lambda^*)|\,ds$$

$$\le\|\emptyset-\emptyset^*\|_{I_0}+L^*\tau|\lambda-\lambda^*|+L^*\int_{t_0}^{t}\|x_s-y_s\|_{I_0}\,ds.$$

Proceeding exactly in the same way as in Theorem 6, we obtain

$$(1.3.7)\qquad \|x-y\|_{I_t}\le 2\|\emptyset-\emptyset^*\|_{I_0}+L^*\tau|\lambda-\lambda^*|+L^*\int_{t_0}^{t}\|x-y\|_{I_s}\,ds,$$

for $t\in[t_0,\beta]$. Hence, by applying Gronwall's lemma to (1.3.7),

$$(1.3.8)\qquad \|x-y\|_{I_t}\le[2\|\emptyset-\emptyset^*\|_{I_0}+L^*\tau|\lambda-\lambda^*|]\,e^{L^*(t-t_0)}$$

for $t\in[t_0,\beta]$. From (1.3.6) and (1.3.8) it follows that

$$|x(t)-y(t)| \leq [1+2e^{L^*\tau}]\|\phi-\phi^*\|_{I_0} + L^*\tau(1+e^{L^*\tau})|\lambda-\lambda^*|.$$

Therefore, choosing δ suitably, it can be shown that $|x(t)-y(t)|<\varepsilon$ for $t\in[t_0,\beta]$. This completes the proof of the theorem.

1.4 EXAMPLES.

In the following examples we take D_H in place of D, where $D_H = \{\psi\in AC(I_0) \mid \|\psi\|_{I_0} < H\}$.

EXAMPLE 1. Consider the following functional-differential equation of neutral type:

$$\frac{dx}{dt} = \int_{-\tau}^{0} K_1(t,\theta)x_t(\theta)d\theta + \int_{-\tau}^{0} K_2(t,\theta)\dot{x}_t(\theta)\, d\theta \tag{1.4.1}$$

for an admissible x.

We assume here that $K_1(t,\theta)$ and $K_2(t,\theta)$ in (1.4.1) are measurable in (t,θ), $t\in(a,b)$ and $\theta\in I_0$ with values in R. Further, $|K_1(t,\theta)| \leq L_1$ and $|K_2(t,\theta)| \leq L_2$ for all $t\in(a,b)$ and $\theta\in I_0$, where L_1 and L_2 are positive constants.

We denote by $f(t,x_t,\dot{x}_t)$ the right-hand side of (1.4.1). It follows from the assumptions made above that $f(t,x_t,\dot{x}_t)$ is measurable as a function of t for each fixed admissible x. Further, for any admissible x, we obtain

$$\begin{aligned}|f(t,x_t,\dot{x}_t)| &\leq L_1 \int_{-\tau}^{0} |x(t+\theta)|d\theta + L_2 \int_{-\tau}^{0} |\dot{x}(t+\theta)|\, d\theta \\ &\leq L_1 \int_{t-\tau}^{t} |x(\theta)|\, d\theta + L_2 \int_{t-\tau}^{t} |\dot{x}(\theta)|\, d\theta\end{aligned}$$

$$\leq L_1 \tau \, \| x \|_{I_t} + L_2 \, \| x \|_{I_t}$$

$$< (L_1 \tau + L_2) \, H \; ,$$

since $x_t \in D_H$. We set $m(t) = (L_1 \tau + L_2)H$. Also, for $\psi, \psi^* \in D_H$ and $t \in (a,b)$, we obtain

$$|f(t,\psi,\dot\psi) - f(t,\psi^*,\dot\psi^*)|$$

$$\leq \int_{-\tau}^{0} |K_1(t,\theta)| \; |\psi(\theta) - \psi^*(\theta)| \, d\theta$$

$$+ \int_{-\tau}^{0} |K_2(t,\theta)| \; |\dot\psi(\theta) - \dot\psi^*(\theta)| \; d\theta$$

$$\leq L_1 \int_{-\tau}^{0} |\psi(\theta) - \psi^*(\theta)| \, d\theta + L_2 \int_{-\tau}^{0} |\dot\psi(\theta) - \dot\psi^*(\theta)| \; d\theta$$

$$\leq L_1 \tau \| \psi - \psi^* \|_{I_0} + L_2 \; \| \psi - \psi^* \|_{I_0}$$

$$\leq L \; \| \psi - \psi^* \|_{I_0} \quad ,$$

where $L = L_1 \tau + L_2$. Hence all the conditions of Theorem 4 are satisfied.

<u>EXAMPLE</u> 2. Consider a nonlinear neutral functional-differential equation of the following type:

$$\text{(1.4.2)} \qquad \frac{dx}{dt} = \left[\int_{-\tau}^{0} K_1(t,\theta) x_t(\theta) d\theta \right]^2 + \left[\int_{-\tau}^{0} K_2(t,\theta) \dot x_t(\theta) d\theta \right]^2$$

for an admissible x, where $K_1(t,\theta)$ and $K_2(t,\theta)$ are same as in example 1.4.1.

Denoting the right-hand side of (1.4.2) by $f(t, x_t, \dot x_t)$, it follows from the assumptions on $K_1(t,\theta)$ and $K_2(t,\theta)$ that

$f(t,x_t,\dot{x}_t)$ is measurable in t for each admissible x. Again, for every admissible x,

$$|f(t,x_t,\dot{x}_t)| \leq L_1^2 \left[\int_{-\tau}^{0} |x(t+\theta)|\,d\theta\right]^2 + L_2^2 \left[\int_{-\tau}^{0} |\dot{x}(t+\theta)|\,d\theta\right]^2$$

$$\leq L_1^2 \tau^2 \|x\|_{I_t}^2 + L_2^2 \|x\|_{I_t}^2$$

$$\leq (L_1^2 \tau^2 + L_2^2)\, H^2 \quad .$$

We set $m(t) = (L_1^2\tau^2 + L_2^2)H^2$. Further, for $\psi, \psi^* \epsilon D_H$, we obtain

$|f(t,\psi,\dot{\psi}) - f(t,\psi^*,\dot{\psi}^*)|$

$$\leq \left| \left[\int_{-\tau}^{0} K_1(t,\theta)\psi(\theta)d\theta\right]^2 - \left[\int_{-\tau}^{0} K_1(t,\theta)\,\psi^*(\theta)d\theta\right]^2 \right|$$

$$+ \left| \left[\int_{-\tau}^{0} K_2(t,\theta)\dot{\psi}(\theta)d\theta\right]^2 - \left[\int_{-\tau}^{0} K_2(t,\theta)\,\dot{\psi}^*(\theta)d\theta\right]^2 \right|$$

$$\leq \left[\int_{-\tau}^{0} |K_1(t,\theta)|\ |\psi(\theta)+\psi^*(\theta)|\,d\theta\right]\left[\int_{-\tau}^{0} |K_1(t,\theta)|\ |\psi(\theta)-\psi^*(\theta)|\,d\theta\right]$$

$$+\left[\int_{-\tau}^{0} |K_2(t,\theta)|\ |\dot{\psi}(\theta)+\dot{\psi}^*(\theta)|\,d\theta\right]\left[\int_{-\tau}^{0} |K_2(t,\theta)|\ |\dot{\psi}(\theta)-\dot{\psi}^*(\theta)|\,d\theta\right]$$

$$\leq (L_1\tau \|\psi+\psi^*\|_{I_o})\ (L_1\tau \|\psi-\psi^*\|_{I_o})$$

$$+ (L_2 \|\psi+\psi^*\|_{I_o})\ (L_2 \|\psi-\psi^*\|_{I_o})$$

$$< 2H\,(L_1^2\tau^2 + L_2^2)\ \|\psi-\psi^*\|_{I_o}$$

$$< L\ \|\psi-\psi^*\|_{I_o} \ ,$$

where $L = 2H\,(L_1^2\tau^2 + L_2^2)$. This proves that the assumptions of Theorem 4 are satisfied.

EXAMPLE 3. We consider

$$(1.4.3) \qquad \frac{dx}{dt} = \int_{-\tau}^{0} d_\theta \eta(t,\theta)\ x_t(\theta) + \int_{-\tau}^{0} K(t,\theta)\ \dot{x}_t(\theta)\ d\theta,$$

for an admissible x, where $\eta(t,\theta)$, defined for $t\in(a,b)$ and $\theta\in I_0$ with values in R, is measurable in (t,θ) and is of bounded variation in θ for each $t\in(a,b)$ and $K(t,\theta)$, defined for $t\in(a,b)$, $\theta\in I_0$ with values in R, is measurable in (t,θ) with $|K(t,\theta)| \leq L_1$, where L_1 is a positive constant. Further, $V(t) \leq L_2$, $t\in(a,b)$, where $V(t) = \overset{0}{\underset{-\tau}{V}} \eta\,(t,\theta)$.

So $f(t,x_t,\dot{x}_t)$ is measurable in t for each admissible x, where the right-hand side of (1.4.3) is denoted by $f(t,x_t,\dot{x}_t)$. Also, for all admissible x, we obtain

$$\begin{aligned}|f(t,x_t,\dot{x}_t)| &\leq \int_{-\tau}^{0} |d_\theta \eta(t,\theta)|\ \ |x(t+\theta)| + \int_{-\tau}^{0} |K(t,\theta)|\ |\dot{x}(t+\theta)|\, d\theta \\ &\leq V(t)\ \|x\|_{I_t} + L_1\ \|x\|_{I_t} \\ &< 2(L_1 + L_2)\ H,\end{aligned}$$

since $x_t\in D_H$. We set $m(t) = 2(L_1+L_2)H$. Further, for $t\in(a,b)$ and $\psi.\psi^*\in D_H$,

$$\begin{aligned}&|f(t,\psi,\dot{\psi}) - f(t,\psi^*,\dot{\psi}^*)| \\ &\leq \int_{-\tau}^{0} |d_\theta \eta\,(t,\theta)|\ \ |\psi(\theta)-\psi^*(\theta)|\, d\theta \\ &\qquad + \int_{-\tau}^{0} |K(t,\theta)|\ \ |\dot{\psi}(\theta) - \dot{\psi}^*(\theta)|\ d\theta \\ &\leq V(t)\ \|\psi-\psi^*\|_{I_0} + L_1\ \|\psi-\psi^*\|_{I_0} \\ &\leq (L_1+L_2)\ \|\psi-\psi^*\|_{I_0} \\ &\leq L\ \|\psi-\psi^*\|_{I_0} \quad ,\end{aligned}$$

where $L = L_1 + L_2$. From this it follows that the conditions of Theorem 4 are satisfied.

The following examples can also be treated in the same manner.

EXAMPLE 4. $\frac{dx}{dt} = \int_{-\tau}^{0} K_1(t,\theta)\ |x_t(\theta)|\ d\theta + \int_{-\tau}^{0} K_2(t,\theta)\ |\dot{x}_t(\theta)|\ d\theta$,

for an admissible x, where K_1 and K_2 are same as in example 1.

EXAMPLE 5. $\frac{dx}{dt} = \int_{-\tau}^{0} d_\theta \eta(t,\theta)\ x_t^2(\theta) + [\int_{-\tau}^{0} K(t,\theta)\ \dot{x}_t(\theta)d\theta]^2$,

where $\eta(t,\theta)$ and $K(t,\theta)$ are same as in example 3.

1.5 NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS WITH INFINITE LAG.

In this section we consider a class of neutral functional differential equations with infinite lag. To obtain results similar to the ones given above for this class of equations we require the following changes:

Let $D_H = \{x \in AC((a,b))\ |\ \|x\|_{(a,b)} < H\}$. For each $t \in (a,b)$, $D_H(t) = \{z|\ z = x|(a,t],\ x \in D_H\}$. We consider the following neutral equation:

(1.5.1) $\frac{dx}{dt} = V\ (t, x(\cdot), \dot{x}(\cdot))$

where, for each $t \in (a,b)$, $V(t,\cdot): D_H(t) \longrightarrow E^n$.

DEFINITION 9. Let $t_o \in (a,b)$ and $\phi \in AC_H((a,t_o])$. An absolutely continuous function $x:(a,t_o+\alpha] \longrightarrow E^n$, $t_o < t_o+\alpha < b$, is said to be a solution of (1.5.1) with initial function ϕ at t_o if

(i) $x \in AC_H((a,t_o+\alpha])$,

(ii) $x(t) = \phi(t)$, $t \in (a,t_o]$,

(iii) $x(t)$ satisfies (1.5.1) for $t \in [t_o, t_o+\alpha]$.

In case $t_o+\alpha = b$, then the interval $(a,t_o+\alpha]$ will be replaced by $(a,t_o+\alpha)$.

For the system (1.5.1), Theorem 4 takes the following form:

<u>THEOREM</u> 10. Suppose that the following conditions are satisfied:

(i) $V(t,x(\cdot), \dot{x}(\cdot))$ is measurable in t for each $x \in D_H$,

(ii) There exists an integrable function $m(t), t\in(a,b)$, such that $|V(t,x(\cdot),\dot{x}(\cdot))| \leq m(t)$, for $t\in(a,b)$ and $x\in D_H$,

(iii) $|V(t,x(\cdot),\dot{x}(\cdot)) - V(t,y(\cdot), \dot{y}(\cdot))| \leq L \, \| x-y\|_{(a,t]}$, for $t\in(a,b)$ and $x,y\in D_H$.

Then, for $t_0\in(a,b)$ and $\phi\in AC_H((a,t_0])$, there exists an $\alpha, 0<\alpha<\tau$, such that the system (1.5.1) admits a unique solution $x(t,t_0,\phi)$ in $[t_0,t_0+\alpha]$ with initial function ϕ at t_0.

The proof of this theorem is similar to that of Theorem 4. Theorems concerning extended existence, continuous dependence can be established for the system (1.5.1) in a way analogous to the earlier treatment.

CHAPTER 2

EXISTENCE OF SOLUTIONS WITHOUT LIPSCHITZ CONDITION

In this chapter an attempt has been made to relax the Lipschitz condition on f by some regularity condition. As a result of this we obtain simple existence of solutions. The inter relationship between the Lipschitz condition and regularity condition has been demonstrated by some remarks. An example has been given to show that the Theorem 4 is not applicable to this where as Theorem 12 can be applied. Also a theorem concerning global existence of solutions has been established.

2.1 EXISTENCE OF SOLUTIONS.

In this section we are considering equation (1.2.3).

DEFINITION 11. Functional $f(t,\psi,\dot{\psi})$ in equation (1.2.3) is said to be continuous in $\psi \in D$ for each $t \in (a,b)$ if given $\varepsilon>0$ there exists a $\delta=\delta(\varepsilon,t)>0$ such that $\|\psi-\psi^*\|_{I_0}<\delta$ implies $|f(t,\psi,\dot{\psi})-f(t,\psi^*,\dot{\psi}^*)|<\varepsilon$, where $\psi^* \in D$. If δ is independent of t then $f(t,\psi,\dot{\psi})$ is said to be continuous in ψ uniformly in $t \in (a,b)$.

THEOREM 12. Suppose that the following conditions are satisfied:

(H_1) $f(t,x_t,\dot{x}_t)$ is measurable in t for each fixed admissible x,

(H_2) There exists an integrable function $m:(a,b) \longrightarrow R^+$ such that $|f(t,x_t,\dot{x}_t)| \leq m(t)$, for all admissible x,

(H_3) $f(t,\psi,\dot{\psi})$ is continuous in $\psi \in D$ for each $t \in (a,b)$,

$$(H_4)\quad \lim_{\sigma\to 0} \int_{\eta_1}^{\eta_2} |f(t+\sigma, x_{t+\sigma}, \dot{x}_{t+\sigma}) - f(t, x_t, \dot{x}_t)|\, dt = 0$$

uniformly with respect to admissible x, where $a<\eta_1<\eta_2<b$ and may be different for different x.

Then, for $t_0 \in (a,b)$ and $\emptyset \in D$, there exists an α, $0<\alpha<\tau$, such that the system (1.2.3) admits a solution $x(t,t_0,\emptyset)$ in $[t_0, t_0+\alpha]$ with initial function $\emptyset$ at t_0.

Note: For $t\in[\eta_1, \eta_2]$, we define

$$f(t+\sigma,\ x_{t+\sigma}, \dot{x}_{t+\sigma}) = 0 \quad , \text{ for } t+\sigma>\eta_2$$

and $$f(t+\sigma,\ x_{t+\sigma}, \dot{x}_{t+\sigma}) = 0 \quad , \text{ for } t+\sigma<\eta_1 .$$

Proof: We define a function $M: [t_0-\tau, b) \longrightarrow R^+$ as in (1.2.5) and a function $\bar{\emptyset}: [t_0-\tau, b) \longrightarrow E^n$ as in (1.2.4).

Since D is open, there exists on $r>0$ such that $\emptyset \in N_{\emptyset}(r) \subset D$. From a discussion in Cahpter 1, it follows that there exists a β_1, $0<\beta_1\le\tau$, such that $\bar{\emptyset}_t \in D$ for $t\in[t_0, t_0+\beta_1]$ and $x_t \in D$ for $x\in\bar{N}_{\bar{\emptyset}}(r^*)$ and $t\in[t_0, t_0+\beta_1]$, where $0<r^*<r/4$.

From the properties of $M(t)$, it is possible to choose an α such that $0<\alpha<\beta_1$ and $M(t_0+\alpha) < r^*$.

We define a sequence of functions $< x^k >$ as follows:

$$(2.1.1)\quad x^k(t) = \begin{cases} \bar{\emptyset}(t), & t\in[t_0-\tau, t_0+\frac{\alpha}{k}] \\ \emptyset(0) + \int_{t_0}^{t-\frac{\alpha}{k}} f(s, x_s^k, \dot{x}_s^k)\,ds, & t\in[t_0+\frac{\alpha}{k}, t_0+\alpha], \end{cases}$$

where $k=1,2,\ldots$.

Now we prove that x^ks are well-defined. Clearly $x^1(t) = \bar{\emptyset}(t)$ and hence is defined. For $k\geq 2$, the first expression of (2.1.1) defines $x^k(t)$ for $t\in[t_0-\tau, t_0+\frac{\alpha}{k}]$. Since $x_t^k = \bar{\emptyset}_t$

and $\bar{\phi}_t \in D$ for $t \in [t_o, t_o + \frac{\alpha}{k}]$, then the second expression of (2.1.1) defines $x^k(t)$ in the interval $[t_o + \frac{\alpha}{k}, t_o + \frac{2\alpha}{k}]$. Hence $x^k(t)$ is defined, as an absolutely continuous function, in $[t_o - \tau, t_o + \frac{2\alpha}{k}]$. Suppose that $x^k(t)$ is defined in $[t_o - \tau, t_o + \frac{j\alpha}{k}]$. We show that $x^k(t)$ is also defined for $t \in [t_o - \tau, t_o + \frac{(j+1)\alpha}{k}]$. For this it is sufficient to show that $x^k_t \in D$ for $t \in [t_o, t_o + \frac{j\alpha}{k}]$. We define a function $\bar{x}^k : [t_o - \tau, t_o + \beta_1] \longrightarrow E^n$ as follows:

$$\bar{x}^k(t) = \begin{cases} x^k(t) & , \ t \in [t_o - \tau, t_o + \frac{j\alpha}{k}] \\ x^k(t_o + \frac{j\alpha}{k}) & , \ t \in [t_o + \frac{j\alpha}{k}, \ t_o + \beta_1] . \end{cases}$$

Clearly $\bar{x}^k$ is an absolutely continuous function. Further,

$$\begin{aligned} \| \bar{x}^k - \bar{\phi} \|_{[t_o - \tau, t_o + \beta_1]} &= |\bar{x}^k(t_o - \tau) - \bar{\phi}(t_o - \tau)| + \int_{t_o - \tau}^{t_o + \beta_1} |\dot{\bar{x}}^k(t) - \dot{\bar{\phi}}(t)| \, dt \\ &= \int_{t_o + \frac{\alpha}{k}}^{t_o + \frac{j\alpha}{k}} | \dot{x}^k(t) | \, dt \\ &= \int_{t_o + \frac{\alpha}{k}}^{t_o + \frac{j\alpha}{k}} | f(t - \tfrac{\alpha}{k}, x^k_{t - \frac{\alpha}{k}}, \dot{x}^k_{t - \frac{\alpha}{k}}) | \, dt \\ &\leq \int_{t_o}^{t_o + \alpha} m(t) \, dt \\ &< r^* . \end{aligned}$$

Hence $\bar{x}^k \in N_{\bar{\phi}}(r^*)$. This proves our assertion. By induction it follows that $x^k(t)$ is defined for $t \in [t_o - \tau, t_o + \alpha]$.

In the following we show that the sequence $\langle x^k \rangle$ is uniformly bounded and

$$\lim_{\sigma\to 0} \int_{t_0-\tau}^{t_0+\alpha} | \dot{x}^k(t+\sigma) - \dot{x}^k(t)| \, dt = 0$$

uniformly with respect to k. We define $< z^k >$ as follows:

$$(2.1.2) \quad z^k(t) = \begin{cases} x^k(t) & , t\in[t_0-\tau, t_0+\alpha] \\ x^k(t_0+\alpha) & , t\in[t_0+\alpha, t_0+\beta_1] \end{cases} .$$

Any easy calculation shows that $\| z^k - \bar{\emptyset}\|_{[t_0-\tau,t_0+\beta_1]} < r^*$ and $\| x^k - \bar{\emptyset}\|_{[t_0-\tau,t_0+\alpha]} = \| z^k - \bar{\emptyset}\|_{[t_0-\tau,t_0+\beta_1]}$. Hence

$\| x^k \|_{[t_0-\tau,t_0+\alpha]} < r^* + \| \emptyset \|_{I_0}$. This proves that the sequence $< x^k >$ is uniformly bounded. Further we see that, for $\sigma>0$,

$$(2.1.3) \quad \int_{t_0-\tau}^{t_0+\alpha} |\dot{x}^k(t+\sigma) - \dot{x}^k(t)| \, dt$$

$$= \int_{t_0-\tau}^{t_0-\sigma} |\dot{x}^k(t+\sigma) - \dot{x}^k(t)| dt + \int_{t_0-\sigma}^{t_0-\sigma+\frac{\alpha}{k}} |\dot{x}^k(t+\sigma) - \dot{x}^k(t)| \, dt$$

$$+ \int_{t_0-\sigma+\frac{\alpha}{k}}^{t_0+\frac{\alpha}{k}} | \dot{x}^k(t+\sigma) - \dot{x}^k(t)| \, dt + \int_{t_0+\frac{\alpha}{k}}^{t_0+\alpha-\sigma} |\dot{x}^k(t+\sigma) - \dot{x}^k(t)| \, dt$$

$$+ \int_{t_0+\alpha-\sigma}^{t_0+\alpha} |\dot{x}^k(t+\sigma) - \dot{x}^k(t)| \, dt$$

$$= I_1 + I_2 + I_3 + I_4 + I_5 .$$

Since the singleton $\{\emptyset\}$ is relatively compact then by Lemma 1 we have

$$I_1 = \int_{t_0-\tau}^{t_0-\sigma} |\dot{\emptyset}(t+\sigma-t_0) - \dot{\emptyset}(t-t_0)| \, dt \le \int_{-\tau}^{0} |\dot{\emptyset}(s+\sigma) - \dot{\emptyset}(s)| \, ds$$

So $I_1 \to 0$ as $\sigma \to 0$ uniformly in k. Again

$$I_2 = \int_{t_0-\sigma}^{t_0-\sigma+\frac{\alpha}{k}} |\dot{\phi}(t+\sigma) - \dot{\phi}(t)|\,dt \le \int_{t_0-\sigma}^{t_0} |\dot{\phi}(t)|\,dt$$

So $I_2 \to 0$ as $\sigma \to 0$ uniformly in k. If $t_0-\sigma+\frac{\alpha}{k} > t_0$, then

$$I_3 = \int_{t_0-\sigma+\frac{\alpha}{k}}^{t_0+\frac{\alpha}{k}} |f(t+\sigma-\tfrac{\alpha}{k},\, x^k_{t+\sigma-\frac{\alpha}{k}},\, \dot{x}^k_{t+\sigma-\frac{\alpha}{k}}) - \dot{\phi}(t)|\,dt$$

$$= \int_{t_0}^{t_0+\sigma} |\, f(s, x^k_s, \dot{x}^k_s)|\,ds$$

$$\le M(t_0+\sigma)\ .$$

Hence $I_3 \to 0$ as $\sigma \to 0$ uniformly in k. If $t_0-\sigma+\frac{\alpha}{k} \le t_0$, then

$$I_3 \le \int_{t_0-\sigma+\frac{\alpha}{k}}^{t_0+\frac{\alpha}{k}} |f(t+\sigma-\tfrac{\alpha}{k},\, x^k_{t+\sigma-\frac{\alpha}{k}},\, \dot{x}^k_{t+\sigma-\frac{\alpha}{k}})|\,dt$$

$$+ \int_{t_0-\sigma+\frac{\alpha}{k}}^{t_0+\frac{\alpha}{k}} |\,\dot{\phi}(t)\,|\,dt$$

$$\le \int_{t_0}^{t_0+\sigma} |\, f(s, x^k_s, \dot{x}^k_s)|\,ds + \int_{t_0-\sigma+\frac{\alpha}{k}}^{t_0+\frac{\alpha}{k}} |\dot{\phi}(t)|\,dt$$

$$\le M\,(t_0+\sigma) + \int_{t_0-\sigma+\frac{\alpha}{k}}^{t_0+\frac{\alpha}{k}} |\,\dot{\phi}(t)|\,dt$$

So $I_3 \to 0$ as $\sigma \to 0$ uniformly with respect to k. Also

$$I_4 = \int_{t_0+\frac{\alpha}{k}}^{t_0+\alpha-\sigma} |f(t+\sigma-\tfrac{\alpha}{k},\, x^k_{t+\sigma-\frac{\alpha}{k}},\, \dot{x}^k_{t+\sigma-\frac{\alpha}{k}}) - f(t-\tfrac{\alpha}{k},\, x^k_{t-\frac{\alpha}{k}},\, \dot{x}^k_{t-\frac{\alpha}{k}})|\,dt$$

$$\le \int_{t_0}^{t_0+\alpha} |f(s+\sigma, x^k_{s+\sigma}, \dot{x}^k_{s+\sigma}) - f(s, x^k_s, \dot{x}^k_s)|\,ds\ .$$

From the assumption (H_4) of the theorem it follows that $I_4 \to 0$ as $\sigma \to 0$ uniformly in k. Lastly,

$$I_5 = \int_{t_o+\alpha-\sigma}^{t_o+\alpha} |f(t-\tfrac{\alpha}{k}, x^k_{t-\frac{\alpha}{k}}, \dot{x}^k_{t-\frac{\alpha}{k}})|\, dt$$

$$\leq \int_{t_o+\alpha-\sigma}^{t_o+\alpha} m(t-\tfrac{\alpha}{k})\, dt$$

$$\leq (M(t_o+\alpha-\tfrac{\alpha}{k}) - M(t_o+\alpha-\tfrac{\alpha}{k}-\sigma)) .$$

So $I_5 \to 0$ as $\sigma \to 0$ uniformly with respect to k. Therefore from (2.1.3) we obtain

$$\int_{t_o-\tau}^{t_o+\alpha} |\dot{x}^k(t+\sigma) - \dot{x}^k(t)|\, dt \to 0 \text{ as } \sigma \to 0 \text{ uniformly in } k.$$

For $\sigma < 0$ we prove our assertion exactly in the above manner.

Now from Lemma 1 it follows that the sequence $\langle x^k \rangle$ is relatively compact and hence it has a subsequence $\langle x^{k_i} \rangle$, say, which converges to an absolutely continuous function $x^o : [t_o-\tau, t_o+\alpha] \longrightarrow E^n$, that is,

(2.1.4) $$\| x^{k_i} - x^o \|_{[t_o-\tau, t_o+\alpha]} \to 0 \text{ as } i \to \infty.$$

Since $\| x^{k_i} - x^o \|_{I_{t_o}} = \| \phi - x^o_{t_o} \|_{I_o}$ and $\| x^{k_i} - x^o \|_{I_{t_o}} \leq \| x^{k_i} - x^o \|_{[t_o-\tau, t_o+\alpha]}$, we have $x^o_{t_o} = \phi$. We define

$z^o : [t_o-\tau, t_o+\beta_1] \longrightarrow E^n$ as

$$z^o(t)=\begin{cases} x^o(t) & , t\in[t_o-\tau, t_o+\alpha] \\ x^o(t_o+\alpha) & , t\in[t_o+\alpha, t_o+\beta_1] \end{cases} .$$

Corresponding to subsequence $\langle x^{k_i} \rangle$, consider the subsequence $\langle z^{k_i} \rangle$ of $\langle z^k \rangle$ defined by (2.1.2). So $\| z^{k_i}-\bar{\emptyset}\|_{[t_o-\tau, t_o+\beta_1]}<r^*$. Since

$$\| z^o-\bar{\emptyset}\|_{[t_o-\tau, t_o+\beta_1]}\leq \| z^{k_i}-\bar{\emptyset}\|_{[t_o-\tau, t_o+\beta_1]}+\|z^{k_i}-z^o\|_{[t_o-\tau, t_o+\beta_1]}$$

$$< r^* + \| x^{k_i}-x^o\|_{[t_o-\tau, t_o+\alpha]},$$

From (2.1.4) it follows that $\| z^o-\bar{\emptyset}\|_{[t_o-\tau, t_o+\beta_1]}\leq r^*$. Hence, $z^o\in\bar{N}_{\bar{\emptyset}}(r^*)$. Consequently, $x_t^o\in D$, for $t\in[t_o, t_o+\alpha]$.

Since for $t\in[t_o, t_o+\alpha]$

$$\| x_t^{k_i} - x_t^o \|_{I_o}\leq 2 \| x^{k_i}- x^o\|_{[t_o-\tau, t_o+\alpha]} ,$$

from assumption (H_3) we obtain

$$| f(t, x_t^{k_i}, \dot{x}_t^{k_i}) - f(t, x_t^o , \dot{x}_t^o)| \to 0$$

as $i \to \infty$ for each fixed $t\in[t_o, t_o+\alpha]$. Hence Lebesgue's dominated convergence theorem implies that

(2.1.5) $$\lim_{i \to \infty} \int_{t_o}^{t} f(s, x_s^{k_i}, \dot{x}_s^{k_i})ds = \int_{t_o}^{t} f(s, x_s^o, \dot{x}_s^o) ds ,$$

for $t\in[t_o, t_o+\alpha]$. Further, since $|x^{k_i}(t)-x^o(t)|\leq\|x^{k_i}-x^o\|_{[t_o-\tau, t_o+\alpha]}$, for $t\in[t_o, t_o+\alpha]$ and

$$\int_{t-\frac{\alpha}{k_i}}^{t} f(s, x_s^{k_i}, \dot{x}_s^{k_i})\, ds \to 0 \text{ as } i \to \infty ,$$

letting $i \to \infty$ and using (2.1.5), it follows from (2.1.1) that

$$x^o(t) = \emptyset(0) + \int_{t_o}^{t} f(s, x_s^o, \dot{x}_s^o)\, ds$$

for $t \in [t_o, t_o+\alpha]$.

Hence $x^o(t)$ is a solution of (1.2.3) with initial function $\emptyset$ at t_o. This completes the proof of the theorem.

2.2 SOME RELATED REMARKS.

REMARK 1. Our Theorem 12 follows if the assumption (H_4) is replaced by

$$(H_4') \quad \lim_{\sigma \to 0} \int_{t_o}^{t_o+\alpha} |f(t+\sigma, x_{t+\sigma}^k, \dot{x}_{t+\sigma}^k) - f(t, x_t^k, \dot{x}_t^k)|\, dt = 0$$

uniformly with respect to k, where 'α' is same as in section 2.1 and $\langle x^k \rangle$ is defined by (2.1.1).

REMARK 2. The proof of Theorem 12 runs smoothly if D is replaced by $N_\emptyset(r)$ in hypotheses (H_1) - (H_4), where $N_\emptyset(r)$ is the same as in section 1.2. It may be noted that D enters into hypotheses (H_1), (H_2) and (H_4) through the definition of admissible functions.

REMARK 3. Suppose that $f:(a,b) \times N_\emptyset(r) \longrightarrow E^n$. Let the assumptions (H_1) and (H_2) hold with D replaced by $N_\emptyset(r)$. If $f(t, \psi, \dot{\psi})$ is Lipschitzian in t and ψ, then (H_4') is satisfied. This can be shown as follows:

$$(2.2.1) \quad \int_{t_o}^{t_o+\alpha} |f(t+\sigma, x^k_{t+\sigma}, \dot{x}^k_{t+\sigma}) - f(t, x^k_t, \dot{x}^k_t)| \, dt$$

$$\leq L \int_{t_o}^{t_o+\alpha} (|\sigma| + \| x^k_{t+\sigma} - x^k_t \|_{I_o}) \, dt$$

$$\leq L |\sigma| \alpha + L \int_{t_o}^{t_o+\alpha} \| x^k_{t+\sigma} - x^k_t \|_{I_o} dt ,$$

where L is the Lipschitz constant.

We show that $\| x^k_{t+\sigma} - x^k_t \|_{I_o} \to 0$ as $\sigma \to 0$ uniformly in k and t. Now, for $t \in [t_o, t_o+\alpha]$,

$$(2.2.2) \quad \|x^k_{t+\sigma} - x^k_t\|_{I_o} = |x^k(t+\sigma-\tau) - x^k(t-\tau)| + \int_{t-\tau}^{t} |\dot{x}^k(\theta+\sigma) - \dot{x}^k(\theta)| \, d\theta .$$

Given $\epsilon > 0$. We choose $0 < \delta < \tau - \alpha$. Let $|\sigma| < \delta$. For $t \in [t_o, t_o+\alpha]$, $|x^k(t+\sigma-\tau) - x^k(t-\tau)| = |\phi(t+\sigma-\tau-t_o) - \phi(t-\tau-t_o)|$. Further, for $t \in [t_o, t_o + \frac{\alpha}{k}]$,

$$(2.2.3) \quad \int_{t-\tau}^{t} |\dot{x}^k(\theta+\sigma) - \dot{x}^k(\theta)| \, d\theta$$

$$\leq \int_{t-\tau}^{t_o+\frac{\alpha}{k}} |\dot{x}^k(\theta+\sigma) - \dot{x}^k(\theta)| \, d\theta$$

$$\leq \int_{t-\tau}^{t_o-\sigma} |\dot{x}^k(\theta+\sigma) - \dot{x}^k(\theta)| d\theta + \int_{t_o-\sigma}^{t_o+\frac{\alpha}{k}-\sigma} |\dot{x}^k(\theta+\sigma) - \dot{x}^k(\theta)| \, d\theta$$

$$+ \int_{t_o+\frac{\alpha}{k}-\sigma}^{t_o+\frac{\alpha}{k}} |\dot{x}^k(\theta+\sigma) - \dot{x}^k(\theta)| \, d\theta$$

$$\leq I_1 + I_2 + I_3 ,$$

where $I_1 = \int_{t-\tau}^{t_0-\sigma} |\dot{\phi}(\theta+\sigma-t_0)-\dot{\phi}(\theta-t_0)|\,d\theta \le \int_{-\tau}^{0} |\dot{\phi}(s+\sigma)-\dot{\phi}(s)|\,ds$,

$$I_2 = \int_{t_0-\sigma}^{t_0+\frac{\alpha}{k}-\sigma} |\dot{\phi}(\theta)|\,d\theta \le \int_{t_0-\sigma}^{t_0} |\dot{\phi}(\theta-t_0)|\,d\theta = \int_{-\sigma}^{0} |\dot{\phi}(s)|\,ds$$

and

$$I_3 \le \int_{t_0+\frac{\alpha}{k}-\sigma}^{t_0+\frac{\alpha}{k}} |\dot{x}^k(\theta+\sigma)|\,d\theta + \int_{t_0+\frac{\alpha}{k}-\sigma}^{t_0+\frac{\alpha}{k}} |\dot{x}^k(\theta)|\,d\theta$$

$$\le \int_{t_0+\frac{\alpha}{k}-\sigma}^{t_0+\frac{\alpha}{k}} |f(\theta+\sigma-\tfrac{\alpha}{k}, x^k_{\theta+\sigma-\frac{\alpha}{k}}, \dot{x}^k_{\theta+\sigma-\frac{\alpha}{k}})|\,d\theta + \int_{t_0-\sigma}^{t_0} |\dot{\phi}(\theta-t_0)|\,d\theta$$

$$\le M(t_0+\sigma) + \int_{-\sigma}^{0} |\dot{\phi}(s)|\,ds$$

Setting $\upsilon(\sigma) = |\phi(t+\sigma-\tau-t_0)-\phi(t-\tau-t_0)| + \int_{-\tau}^{0} |\dot{\phi}(s+\sigma)-\dot{\phi}(s)|\,ds$

$$+ M(t_0+\sigma) + 2\int_{-\sigma}^{0} |\dot{\phi}(s)|\,ds$$

and taking into account the fact that ϕ is uniformly continuous, singleton $\{\phi\}$ is relatively compact, $\dot{\phi}$ is integrable and $M(t)$ is continuous, one can see that $\upsilon(\sigma) \to 0$ as $\sigma \to 0$ uniformly in k and t. Hence, for $t \in [t_0, t_0+\frac{\alpha}{k}]$, we obtain from (2.2.2) that $|\sigma| < \delta$ implies $\| x^k_{t+\sigma} - x^k_t \|_{I_0} < \varepsilon$. Also, for $t \in (t_0+\frac{\alpha}{k}, t_0+\alpha]$,

(2.2.4) $$\int_{t-\tau}^{t} |\dot{x}^k(\theta+\sigma) - \dot{x}^k(\theta)|\,d\theta$$

$$= \int_{t-\tau}^{t_0-\sigma} |\dot{x}^k(\theta+\sigma)-\dot{x}^k(\theta)|\,d\theta + \int_{t_0-\sigma}^{t_0-\sigma+\frac{\alpha}{k}} |\dot{x}^k(\theta+\sigma)-\dot{x}^k(\theta)|\,d\theta$$

$$+ \int_{t_0-\sigma+\frac{\alpha}{k}}^{t_0+\frac{\alpha}{k}} |\dot{x}^k(\theta+\sigma)-\dot{x}^k(\theta)|\,d\theta + \int_{t_0+\frac{\alpha}{k}}^{t} |\dot{x}^k(\theta+\sigma)-\dot{x}^k(\theta)|\,d\theta$$

But first three integrals in right-hand side of (2.2.4) are same as right-hand side of (2.2.3) and last integral is given by:

$$(2.2.5)\quad \int_{t_0+\frac{\alpha}{k}}^{t} | \dot{x}^k(\theta+\sigma) - \dot{x}^k(\theta)| \, d\theta$$

$$= \int_{t_0+\frac{\alpha}{k}}^{t} | f(\theta+\sigma-\tfrac{\alpha}{k}, x^k_{\theta+\sigma-\frac{\alpha}{k}}, \dot{x}^k_{\theta+\sigma-\frac{\alpha}{k}}) - f(\theta-\tfrac{\alpha}{k}, x^k_{\theta-\frac{\alpha}{k}}, \dot{x}^k_{\theta-\frac{\alpha}{k}})| \, d\theta$$

$$\leq \int_{t_0}^{t} | f(s+\sigma, x^k_{s+\sigma}, \dot{x}^k_{s+\sigma}) - f(s, x^k_s, \dot{x}^k_s)| \, ds$$

$$\leq L\,\alpha\,|\sigma| + L \int_{t_0}^{t} \| x^k_{s+\sigma} - x^k_s \|_{I_0} \, ds \,.$$

From (2.2.2), (2.2.4) and (2.2.5) it follows that

$$(2.2.6)\qquad \| x^k_{t+\sigma} - x^k_t \|_{I_0} \leq \upsilon(\sigma) + L\,\alpha\,|\sigma| + L \int_{t_0}^{t} \| x^k_{s+\sigma} - x^k_s \|_{I_0} \, ds \,.$$

Consequently, by applying Gronwall's lemma to (2.2.6), we obtain

$$(2.2.7)\qquad \| x^k_{t+\sigma} - x^k_t \|_{I_0} \leq \upsilon^*(\sigma)\, e^{L(t-t_0)}$$

$$\leq \upsilon^*(\sigma)\, e^{L\alpha} \,,$$

where $\upsilon^*(\sigma) = \upsilon(\sigma) + L\,\alpha\,|\sigma|$. Clearly $\upsilon^*(\sigma) \to 0$ as $\sigma \to 0$ uniformly in k and t. Hence from (2.2.7) we get $\| x^k_{t+\sigma} - x^k_t \|_{I_0} < \varepsilon$ whenever $|\sigma| < \delta$, that is, for $t \in (t_0 + \frac{\alpha}{k}, t_0+\alpha]$, $\| x^k_{t+\sigma} - x^k_t \|_{I_0} \to 0$ as $\sigma \to 0$ uniformly in k. Consequently, for $t \in [t_0, t_0+\alpha]$,

$$\| x^k_{t+\sigma} - x^k_t \|_{I_0} \to 0 \text{ as } \sigma \to 0 \text{ uniformly in } k.$$

From (2.2.1) it follows that

$$\int_{t_o}^{t_o+\alpha} |f(t+\sigma,\ x^k_{t+\sigma}, \dot{x}^k_{t+\sigma}) - f(t, x^k_t,\ \dot{x}^k_t\ )|\ dt \to 0 \text{ as } \sigma \to 0$$

uniformly in k. This proves our assertion.

As a corollary to the last remark, we see that if the equation (1.2.3) is autonomous, then Lipschitz property in ψ implies the condition (H_4').

Now we consider an example to show that even though the right-hand side of the equation satisfies assumptions (H_1) - (H_3) and (H_4'), it is not Lipschitzian. This also by the way proves that there are examples where Theorem 4 is not applicable but the present existence theorem is applicable.

<u>EXAMPLE</u> We consider

$$\frac{dx}{dt} = \int_{t-1}^{t} |\dot{x}(s)|\ ds + \{x(t)\}^{1/2} \quad . \tag{2.2.8}$$

Here we choose a=0, b=1, t_o=0, τ=1 and $\emptyset(t) = 0$ for $t\in[-1,0]$. We define $\bar{\emptyset}$ as in (1.2.4). So $N_\emptyset(r)=\{\psi\in AC([-1,0],R^+) \mid \|\psi\|_{I_o}<r\}$. By an admissible function we mean an absolutely continuous function $x:[\eta_1 -1\ \eta_2] \longrightarrow R^+$, where $0 \leq \eta_1 < \eta_2 \leq 1$, such that $x_t \in N_\emptyset(r)$ for $t\in[\eta_1, \eta_2]$. The right-hand side of (2.2.8) is denoted by $f(t,x_t,\dot{x}_t)$ where $f: [0,1] x\ N_\emptyset(r) \longrightarrow R$.

Clearly $\bar{\emptyset}_t \in N_\emptyset(r)$ for $t\in[0,1]$. So here β_1=1. Further $N_{\bar{\emptyset}}(r^*) = \{x\in AC([-1,1],R^+) \mid \|x\|_{[-1,1]} < r^*\}$, where $0<r^*< \frac{r}{4}$. From the discussion in section 1.2 it follows that $x\in\bar{N}_{\bar{\emptyset}}(r^*)$ is admissible.

Now we show that the assumptions (H_1), (H_2), (H_3) and (H_4') are satisfied. It is easy to see that $f(t,x_t,\dot{x}_t)$ is a measurable function in t for each fixed admissible x. Also, since $|x(t)| \leq \|x_t\|_{I_0}$, we obtain, for all admissible x,

$$|f(t,x_t,\dot{x}_t)| \leq \|x_t\|_{I_0} + \{\|x_t\|_{I_0}\}^{1/2}$$

$$< r + r^{1/2}$$

So $m(t) = r+r^{1/2}$. Further, for $\psi, \psi^* \in N_\emptyset(r)$ and $t \in [0,1]$,

$$|f(t,\psi,\dot{\psi}) - f(t,\psi^*,\dot{\psi}^*)|$$

$$\leq \int_{-1}^{0} \big|\, |\dot{\psi}(s)| - |\dot{\psi}^*(s)| \,\big|\, ds + |\{\psi(0)\}^{1/2} - \{\psi^*(0)\}^{1/2}|$$

$$\leq \int_{-1}^{0} |\dot{\psi}(s) - \dot{\psi}^*(s)|\, ds + |\psi(0) - \psi^*(0)|^{1/2}$$

$$\leq \|\psi-\psi^*\|_{I_0} + \{\|\psi-\psi^*\|_{I_0}\}^{1/2} .$$

Hence $\|\psi-\psi^*\|_{I_0} < \delta$ implies that $|f(t,\psi,\dot{\psi})-f(t,\psi^*,\dot{\psi}^*)| < \delta+\delta^{1/2}$ whence the continuity of $f(t,\psi,\dot{\psi})$ in $\psi \in N_\emptyset(r)$ for $t \in [0,1]$ follows immediately. Next, for $t \in [0,\alpha]$ with $\alpha < 1$, we have

$$|f(t+\sigma,x^k_{t+\sigma},\dot{x}^k_{t+\sigma}) - f(t,x^k_t,\dot{x}^k_t)|$$

$$= \left| \int_{t+\sigma-1}^{t+\sigma} |\dot{x}^k(s)|\, ds + \{x^k(t+\sigma)\}^{1/2} - \int_{t-1}^{t} |\dot{x}^k(s)|\, ds - \{x^k(t)\}^{1/2} \right|$$

$$= \left| \int_{0}^{t+\sigma} |\dot{x}^k(s)|\, ds + \{x^k(t+\sigma)\}^{1/2} - \int_{0}^{t} |\dot{x}^k(s)|\, ds - \{x^k(t)\}^{1/2} \right|$$

$$\leq \left| \int_{t}^{t+\sigma} |\dot{x}^k(s)|\, ds \right| + \left| \{x^k(t+\sigma)\}^{1/2} - \{x^k(t)\}^{1/2} \right|$$

$$\leq \left| \int_t^{t+\sigma} m(s-\tfrac{\alpha}{k})ds \right| + |x^k(t+\sigma) - x^k(t)|^{1/2}$$

$$\leq | M(t+\sigma-\tfrac{\alpha}{k}) - M(t-\tfrac{\alpha}{k})| + |M(t+\sigma-\tfrac{\alpha}{k}) - M(t-\tfrac{\alpha}{k})|^{1/2}.$$

Since $M(t)$ is uniformly continuous, then

$$|f(t+\sigma, x^k_{t+\sigma}, \dot{x}^k_{t+\sigma}) - f(t, x^k_t, \dot{x}^k_t)| \to 0 \text{ as } \sigma \to 0$$

uniformly in t and k. Consequently

$$\lim_{\sigma \to 0} \int_0^{\alpha} |f(t+\sigma, x^k_{t+\sigma}, \dot{x}^k_{t+\sigma}) - f(t, x^k_t, \dot{x}^k_t)| \, dt = 0$$

uniformly in k. This is nothing but (H_4'). Hence f satisfie conditions (H_1), (H_2), (H_3) and (H_4').

In the following paragraph we prove that $f(t, \psi, \dot{\psi})$ is not Lipschitzian in ψ notwithstanding its so many good properties enumerated above. In fact, in order that $f(t, \psi, \dot{\psi})$ satisfies a Lipschitz condition in ψ there should exist an $L > 0$ satisfying

$$|f(t,\psi,\dot{\psi}) - f(t,\psi^*,\dot{\psi}^*)| \leq L \, ||\psi - \psi^*||_{I_0}$$

for $\psi, \psi^* \in N_\emptyset(r)$ and $t \in [0,1]$. In particular, if $\psi^* \equiv 0$, we should have $|f(t, \psi, \dot{\psi})| \leq L \, ||\psi||_{I_0}$. We now construct the following absolutely continuous function defined over $[-1,1]$:

$$x_m(t) = \begin{cases} 0 & , t \in [-1,0] \\ \frac{1}{m^2} - (t-\frac{1}{m})^2 & , t \in [0, \frac{2}{m}] \\ 0 & , t \in [\frac{2}{m}, 1] \end{cases} ,$$

for $m \geq 2$. Here

$$\| x_m \|_{[-1,1]} = |x_m(-1)| + \int_{-1}^{1} |\dot{x}_m(t)| dt$$

$$= \int_0^{2/m} |\dot{x}_m(t)| dt$$

$$= \int_0^{1/m} 2(\tfrac{1}{m} - t) dt + \int_{1/m}^{2/m} 2(t - \tfrac{1}{m}) dt$$

$$= \frac{2}{m^2} .$$

Hence, for sufficiently large m, $\| x_m \|_{[-1,1]}$ can be made less than any preassigned value r*, whence $x_m \in N_{\bar{\emptyset}}(r^*)$. Further

$$\| x_m \|_{I_{\frac{1}{m}}} = |x_m(\tfrac{1}{m} - 1)| + \int_{\frac{1}{m}-1}^{1/m} |\dot{x}_m(\theta)| d\theta$$

$$= \int_0^{1/m} |\dot{x}_m(\theta)| d\theta$$

$$= \frac{1}{m^2}$$

and $\left| \int_{\frac{1}{m}-1}^{1/m} |\dot{x}_m(s)| ds + \{x_m(\tfrac{1}{m})\}^{1/2} \right| = \frac{1}{m^2} + \frac{1}{m} = \frac{m+1}{m^2}$.

Therefore, in order that $|f(t, \psi, \dot{\psi})| \leq L \| \psi \|_{I_0}$ for $\psi = (x_m)_{\frac{1}{m}}$, we should have $\frac{m+1}{m^2} \leq L \frac{1}{m^2}$. But this is impossible for sufficiently large m. This shows that $f(t, \psi, \dot{\psi})$ is not Lipschitzian in ψ .

2.3. UNIQUENESS.

THEOREM 13. Suppose that the conditions (H_1), (H_2), (H_4) of Theorem 12 are satisfied. Further let f satisfy the following condition:

(H_5) $\quad |f(t,\psi,\dot{\psi}) - f(t,\psi^*,\dot{\psi}^*)| \leq g_1(t, \|\psi-\psi^*\|_{I_0})$

for $\psi, \psi^* \in D$ and $t \in (a,b)$, where $g_1 : (a,b) \times R^+ \longrightarrow R^+$ is a continuous function and $g_1(t,u)$ is monotonic increasing in u for each $t \in (a,b)$ and further for each $t_0 \in (a,b)$ and $t_1 \in [t_0,b)$, $u(t) \equiv 0$ is the only differentiable function which satisfies

$$(2.3.1) \qquad \frac{du}{dt} = g_1(t,u) \ , \ u(t_0) = 0$$

for $t \in [t_0,t_1)$.

Then, for $t_0 \in (a,b)$ and $\emptyset \in D$, there exists an α, $0<\alpha<\tau$, such that the system (1.2.3) admits a unique solution $x(t,t_0,\emptyset)$ in $[t_0,t_0+\alpha]$ with initial function $\emptyset$ at t_0.

Proof: The existence of a solution in $[t_0,t_0+\alpha]$ follows from Theorem 12. Let $x(t) = x(t,t_0,\emptyset)$ and $y(t) = y(t,t_0,\emptyset)$ be any two solutions of (1.2.3) in $[t_0,t_0+\alpha]$ with the same initial function $\emptyset$ at t_0. To show that $x(t) \equiv y(t)$. Set $z(t)=x(t)-y(t)$. So $z(t) = 0$ for $t \in [t_0-\tau,t_0]$. Further, for $t>t_0$,

$$\begin{aligned} \|z_t\|_{I_0} &\leq \int_{t_0}^{t} |f(s,x_s,\dot{x}_s) - f(s,y_s,\dot{y}_s)| \, ds \\ &\leq \int_{t_0}^{t} g_1(s, \|x_s-y_s\|_{I_0}) ds \\ &\leq \int_{t_0}^{t} g_1(s, \|z_s\|_{I_0}) \, ds \ . \end{aligned}$$

If $u(t,t_0,u_0)$ is the maximal solution of (2.3.1), then

$\|z_t\|_{I_0} \leq u(t,t_0,u_0)$, $t \in (t_0,t_0+\alpha]$. But from the hypothesis

of the theorem it follows that $u(t,t_o,u_o) = 0$ and hence $\|z_t\|_{I_o} = 0$. Since $|z(t)| \leq \|z_t\|_{I_o}$ for $t \in [t_o, t_o+\alpha]$, then $z(t) = 0$ for $t \in [t_o, t_o+\alpha]$. This completes the proof of the theorem.

2.4 GLOBAL EXISTENCE.

In this section we consider the following neutral functional-differential equation:

(2.4.1) $\quad \frac{dx}{dt} = f(t, x_t, \dot{x}_t)$,

where $f:(a,\infty) \times AC_H(I_o) \longrightarrow E^n$ and $AC_H(I_o) = \{\psi \in AC(I_o) \mid \|\psi\|_{I_o} < H\}$.
By an admissible function we mean an absolutely continuous function $x: [\eta_1 - \tau, \eta_2] \longrightarrow E^n$ such that $x_t \in AC_H(I_o)$ for $t \in [\eta_1, \eta_2]$, where $a < \eta_1 < \eta_2 < \infty$.

We will use the following form of Schauder's fixed point theorem in proving the global existence of solutions of (2.4.1).

THEOREM 14. Let X be a Banach space and B be a closed, bounded, convex subset of X. If a mapping $T: B \longrightarrow X$ is continuous with $T(B) \subset B$ and $T(B)$ relatively compact, then T has a fixed point in B.

THEOREM 15. Suppose that the following conditions are satisfied:

(i) For every $t_o \in (a,\infty)$ and $u_o \in (0, H_1]$, $H_1 < H$, the scalar differential equation

(2.4.2) $\quad \frac{du}{dt} = g_1(t,u)$, $u(t_o) = u_o$

has a solution $u(t) = u(t,t_0,u_0)$ existing for $t\in[t_0,\infty)$ and $u(t) \leq H_1$, for $t\in[t_0,\infty)$, where $g_1:(a,\infty) \times R^+ \longrightarrow R^+$ is a continuous function and $g_1(t,u)$ is monotonic increasing in u for each $t\in(a,\infty)$,

(ii) $f(t,x_t,\dot{x}_t)$ is measurable in t for each fixed admissible x,

(iii) $|f(t,x_t,\dot{x}_t)| \leq \frac{1}{2} g_1(t,\|x_t\|_{I_0})$ for all admissible x,

(iv) $f(t,\psi,\dot{\psi})$ is continuous in $\psi \in AC_H(I_0)$ uniformly with respect to $t\in(a,\infty)$,

(v) $$\lim_{\sigma\to 0} \int_{\eta_1}^{\eta_2} | f(t+\sigma,x_{t+\sigma},\dot{x}_{t+\sigma}) - f(t,x_t,\dot{x}_t)| \, dt = 0$$

uniformly with respect to admissible x, for any η_1, η_2 lying in (a,∞).

Then, for every $t_0\in(a,\infty)$ and $\phi\in AC_H(I_0)$ with $\|\phi\|_{I_0} \leq \frac{1}{2} u_0$, the system (2.4.1) admits a solution $x(t,t_0,\phi)$ existing for $t\in[t_0,t_0+\gamma]$, where γ is any number greater than zero and $\|x_t(t_0,\phi)\|_{I_0} \leq H_1$ for $t\in[t_0,t_0+\gamma]$.

Proof: Let B be a set of absolutely continuous functions $x: [t_0-\tau,t_0+\gamma] \longrightarrow E^n$ such that $x_{t_0} = \phi$, $\|x_t\|_{I_0} \leq u(t)$, for $t\in[t_0,t_0+\gamma]$, and $\|x\|_{[t_0-\tau,t_0+\gamma]} \leq H_1$. To show that B is a closed, bounded, convex subset of $AC([t_0-\tau,t_0+\gamma])$. Clearly B is bounded. Let x^0 be a limit point of B. So there exists a sequence $\langle x^n \rangle$ in B such that $\|x^n-x^0\|_{[t_0-\tau,t_0+\gamma]} \to 0$ as $n\to\infty$. Since $\|x^n_{t_0}-x^0_{t_0}\|_{I_0} \leq \|x^n-x^0\|_{[t_0-\tau,t_0+\gamma]}$, then $x^0_{t_0} = \phi$. It is easy to see that $\|x^0_t\|_{I_0} \leq u(t)$ for $t\in[t_0,t_0+\gamma]$ and

$\| x^o \|_{[t_o-\tau, t_o+\gamma]} \leq H_1$. Hence $x^o \in B$. Consequently B is closed. Set $z = \mu x + (1-\mu)y$, where $x, y \in B$ and $0 \leq \mu \leq 1$. So

$z_{t_o} = \mu x_{t_o} + (1-\mu) y_{t_o} = \mu \phi + (1-\mu)\ \phi = \phi$, $\| z_t \|_{I_o} \leq \mu \| x_t \|_{I_o} + (1-\mu) \| y_t \|_{I_o} \leq u(t)$, and $\| z \|_{[t_o-\tau, t_o+\gamma]} \leq \mu \|x\|_{[t_o-\tau, t_o+\gamma]} + (1-\mu) \| y \|_{[t_o-\tau, t_o+\gamma]} \leq H_1$. This proves our assertion.

From the definition of B it follows that the elements of B are admissible.

Now we define a mapping $T: B \longrightarrow AC([t_o-\tau, t_o+\gamma])$ as follows:

$$
(2.4.3) \qquad
\begin{aligned}
&(Tx)_{t_o} = \phi \\
&Tx(t) = \phi(0) + \int_{t_o}^{t} f(s, x_s, \dot{x}_s)\,ds, \quad t \in [t_o, t_o+\gamma].
\end{aligned}
$$

Continuity of the mapping T follows from the assumption (iv).

In the following we show that $T(B) \subset B$. Let $y \in T(B)$. Then $y = Tx$, for $x \in B$. From (2.4.3) it is clear that $y_{t_o} = \phi$. Now, for $t \in [t_o, t_o+\gamma]$, if $t-\tau \leq t_o$, then

$$
\begin{aligned}
\| y_t \|_{I_o} &= |y(t-\tau)| + \int_{t-\tau}^{t} |\dot{y}(s)|\,ds \\
&\leq |\phi(t-\tau-t_o)| + \int_{t_o-\tau}^{t_o} |\dot{\phi}(s-t_o)|\,ds + \int_{t_o}^{t} |\dot{y}(s)|\,ds \\
&\leq 2 \| \phi \|_{I_o} + \frac{1}{2} \int_{t_o}^{t} g_1(s, \| x_s \|_{I_o})\,ds \\
&\leq u_o + \int_{t_o}^{t} g_1(s, u(s))\,ds \\
&\leq u(t) .
\end{aligned}
$$

If $t-\tau > t_o$, then

$$\|y_t\|_{I_o} \leq |\emptyset(0)| + 2\int_{t_o}^{t} f(s,x_s,\dot{x}_s)\,ds$$
$$\leq \|\emptyset\|_{I_o} + \int_{t_o}^{t} g_1(s,\|x_s\|_{I_o})ds$$
$$\leq u_o + \int_{t_o}^{t} g_1(s,u(s))ds$$
$$\leq u(t) \;,$$

Hence $\|y_t\|_{I_o} \leq u(t)$, for $t \in [t_o, t_o+\gamma]$. Further ,

$$\|y\|_{[t_o-\tau,t_o+\gamma]} = |\emptyset(-\tau)| + \int_{t_o-\tau}^{t_o} |\dot{\emptyset}(s-t_o)|\,ds + \int_{t_o}^{t_o+\gamma} |f(s,x_s,\dot{x}_s)|\,ds$$
$$\leq \|\emptyset\|_{I_o} + \frac{1}{2}\int_{t_o}^{t_o+\gamma} g_1(s,\|x_s\|_{I_o})\,ds$$
$$\leq u_o + \int_{t_o}^{t_o+\gamma} g_1(s,u(s))ds$$
$$\leq u(t_o+\gamma)$$
$$\leq H_1 \;.$$

This proves that $y \in B$.

What remains to show is that $T(B)$ is relatively compact. Obviously $T(B)$ is uniformly bounded. Again

$$(2.4.4)\qquad \int_{t_o-\tau}^{t_o+\gamma} |\dot{y}(t+\sigma) - \dot{y}(t)|\,dt$$
$$= \int_{t_o-\tau}^{t_o-\sigma} |\dot{y}(t+\sigma) - \dot{y}(t)|\,dt + \int_{t_o-\sigma}^{t_o} |\dot{y}(t+\sigma) - \dot{y}(t)|\,dt$$
$$+ \int_{t_o}^{t_o+\gamma} |\dot{y}(t+\sigma) - \dot{y}(t)|\,dt$$
$$= I_1 + I_2 + I_3 \;,$$

where $I_1 = \int_{t_o-\tau}^{t_o-\sigma} |\dot{\phi}(t+\sigma-t_o) - \dot{\phi}(t-t_o)|\, dt \le \int_{-\tau}^{0} |\dot{\phi}(\theta+\sigma) - \dot{\phi}(\theta)|\, d\theta$,

$$I_2 = \int_{t_o-\sigma}^{t_o} |f(t+\sigma, x_{t+\sigma}, \dot{x}_{t+\sigma}) - \dot{\phi}(t-t_o)|\, dt$$

$$\le \frac{1}{2} \int_{t_o-\sigma}^{t_o} g_1(t+\sigma, \| x_{t+\sigma} \|_{I_o})dt + \int_{-\sigma}^{0} |\dot{\phi}(\theta)|\, d\theta$$

$$\le \frac{1}{2} \int_{t_o}^{t_o+\sigma} g_1(s, u(s))ds + \int_{-\sigma}^{0} |\dot{\phi}(\theta)|\, d\theta$$

and $I_3 = \int_{t_o}^{t_o+\gamma} |f(t+\sigma, x_{t+\sigma}, \dot{x}_{t+\sigma}) - f(t, x_t, \dot{x}_t)|\, dt$.

Since singleton $\{\phi\}$ is relatively compact, then making use of the assumption (v) we obtain from (2.4.4) that

$$\lim_{\sigma \to 0} \int_{t_o-\tau}^{t_o+\gamma} |\dot{y}(t+\sigma) - \dot{y}(t)|\, dt = 0$$

uniformly in y. Hence lemma 1 implies that T(B) is relatively compact.

From Theorem 14 it follows that the mapping T has a fixed point in B. Let it be x*. So we obtain

$$x^*_{t_o} = \phi$$

$$x^*(t) = \phi(0) + \int_{t_o}^{t} f(s, x^*_s, \dot{x}^*_s)\, ds, \quad t \in [t_o, t_o+\gamma].$$

Further, from the definition of B it follows that $\|x^*_t(t_o, \phi)\|_{I_o} \le H_1$ for $t \in [t_o, t_o+\gamma]$. This completes the proof of the theorem.

THEOREM 16. Suppose that the conditions (i), (ii),(iii) or (v) of Theorem 15 are satisfied. Further let

(iv)' $\quad |f(t, \psi, \dot{\psi}) - f(t, \psi^*, \dot{\psi}^*)| \leq L \|\psi - \psi^*\|_{I_0}$

for $t \in (a, \infty)$ and $\psi, \psi^* \in {}_{\Delta}C_H(I_0)$, where L is a positive constant

Then, for every $t_0 \in (a, \infty)$ and $\emptyset \in {}_{\Delta}C_H(I_0)$ with $\|\emptyset\|_{I_0} \leq \frac{1}{2} u_0$, the system (2.4.1) admits a unique solution $x(t, t_0, \emptyset)$ existing for $t \in [t_0, t_0 + \gamma]$, where $t_0 < t_0 + \gamma < \infty$.

Proof: Clearly (iv)' implies (iv). So the existence of a solution $x(t, t_0, \emptyset)$ in $[t_0, t_0 + \gamma]$ follows from Theorem 15. Proceeding exactly in the same way as in Theorem 4, it is easy to prove that the solution $x(t, t_0, \emptyset)$ is unique.

COROLLARY 17. Since Theorem 16 holds for every $t_0 + \gamma < \infty$, then we say that the system (2.4.1) admits a unique solution $x(t, t_0, \emptyset)$ in $[t_0, \infty)$.

CHAPTER 3

REPRESENTATION OF SOLUTIONS OF LINEAR EQUATIONS

This chapter is devoted to the representation of solutions of linear non-homogeneous functional-differential equations of neutral type. Theorems on the existence of solutions of these equations and adjoints to homogeneous equations have been established.

3.1 EXISTENCE OF SOLUTIONS.

We consider the following neutral functional-differential equations:

(3.1.1) $$\frac{dx}{dt} = f(t,x_t,\dot{x}_t) + h(t) ,$$

where $f:(a,\infty) \times AC(I_0) \longrightarrow E^n$ and $h \in L_1((a,\infty))$.

DEFINITION 18. Functional $f(t,\psi,\dot{\psi})$ is said to be linear in ψ if

(i) $f(t, \psi + \psi^*, \dot{\psi} + \dot{\psi}^*) = f(t, \psi , \dot{\psi}) + f(t,\psi^* , \dot{\psi}^*)$, where $t \in (a,\infty)$ and $\psi,\psi^* \in AC(I_0)$,

(ii) For any real number μ, $f(t, \mu\psi , \mu\dot{\psi}) = \mu f(t, \psi,\dot{\psi})$.

Suppose that $f(t,\psi,\dot{\psi})$ is linear in ψ and continuous in ψ for each fixed t in (a,∞). So $f(t,\psi,\dot{\psi})$ can be written as:

(3.1.2) $$f(t,\psi,\dot{\psi}) = \mu\psi(c) + \int_{-\tau}^{0} g(t,\theta)\, \dot{\psi}(\theta)\, d\theta ,$$

where $\mu \in R$, $c \in I_o$ and $g:(a,\infty) \times I_o \longrightarrow E^{n^2}$ such that for each $t \in (a,\infty)$, $g(t,.) \in L_\infty(I_o, E^{n^2})$.

In this chapter, an admissible function x means an absolutely continuous function $x:[\eta_1-\tau, \eta_2] \longrightarrow E^n$, where $a<\eta_1<\eta_2<\infty$. So, for an admissible x, (3.1.1) takes the form

$$\frac{dx}{dt} = \mu x(t+c) + \int_{-\tau}^{0} g(t,\theta)\, \dot{x}(t+\theta)\, d\theta + h(t). \tag{3.1.3}$$

The homogeneous system corresponding to the system (3.1.3) is given by

$$\frac{dx}{dt} = \mu x(t+c) + \int_{-\tau}^{0} g(t,\theta)\, \dot{x}(t+\theta)\, d\theta. \tag{3.1.4}$$

We group below the set of all conditions that are imposed on g in different theorems that follow:

(A_1) $g(t,\theta)$ is measurable in (t,θ),

(A_2) $|g(t,\theta)| \leq (\frac{1}{2} m(t) - |\mu|)$, for $t \in (a,\infty)$ and $\theta \in I_o$, where $m \in L_1((a,\infty), R^+)$,

(A_3) $g(t,\theta)$ is essentially bounded in t for each fixed θ and for each t,

$$|g(t,\theta_1) - g(t,\theta_2)| \leq N(t)\, |\theta_1 - \theta_2| ,$$

where $N: (a,\infty) \longrightarrow R^+$ is integrable.

(A_4) $|g(t,\theta)| \leq (\frac{1}{2} m_1(t) - |\mu|)$, for $t \in (a,\infty)$ and $\theta \in I_o$, where $m_1 \in L_1((a,\infty), R^+)$, such that $N^*(t_o,\beta)\beta + 3M^*(t_o,\beta) + 2|\mu|\beta \neq 1$, where

$$M^*(t_o,\beta) = \int_{t_o}^{t_o+\beta+\tau} (\tfrac{1}{2} m_1(t) - |\mu|)dt \text{ and } N^*(t_o,\beta) = \int_{t_o}^{t_o+\beta+\tau} N(s)ds,$$

$t_0 \in (a,\infty)$ and $\beta \in (a,\infty)$,

(A_5) (i) $g(t,\theta) = 0$, for $t \in (a,\infty)$, $\theta < -\tau$ and

(ii) $g(t,\theta) = 0$, for $t \in (a,\infty)$, $\theta > 0$,

(A_6) $\lim\limits_{\sigma \to 0} \int_a^\infty |g(t+\sigma,0) - g(t,0)| \, dt = 0$

(A_7) $\lim\limits_{\sigma \to 0} \int_a^\infty \left| \frac{\partial g(t,\theta+\sigma)}{\partial\theta} - \frac{\partial g(t,\theta)}{\partial\theta} \right| dt = 0$ uniformly in θ.

THEOREM 19. Let the assumptions (A_1) and (A_2) be satisfied. Further suppose that

(A_8) $|h(t)| \leq \frac{1}{2} m(t)$, for $t \in (a,\infty)$,

where $m(t)$ is the same as in (A_2).

Then, for $t_0 \in (a,\infty)$ and $\phi \in AC(I_0)$, the system (3.1.3) admits a unique solution $x(t,t_0,\phi,h)$ in $[t_0,t_0+\beta]$ with initial function ϕ at t_0, where $t_0 < t_0 + \beta < \infty$.

Proof. We define a function $M: [t_0-\tau,\infty) \longrightarrow R^+$ as in (1.2.5) and a function $\bar{\phi}: [t_0-\tau,\infty) \longrightarrow E^n$ as in (1.2.4). Further we define a sequence $\langle x^k \rangle$ as follows:

$$x^0(t) = \bar{\phi}(t), \quad t \in [t_0-\tau, t_0+\beta]$$

$$x^k_{t_0}(\theta) = \phi(\theta), \quad \theta \in I_0$$

(3.1.5)
$$x^k(t) = \phi(0) + \mu \int_{t_0}^{t} x^{k-1}(s+c)\,ds + \int_{t_0}^{t} \left[\int_{-\tau}^{0} g(s,\theta)\, x^{k-1}(s+\theta)\,d\theta\right] ds + \int_{t_0}^{t} h(s)\,ds \,, \quad t \in [t_0,t_0+\beta] ,$$

where $k=1,2,3,\ldots$

Clearly $x^1(t)$ is well-defined provided $\int_{-\tau}^{0} g(s,\theta)\dot{x}^0(s+\theta)d\theta$ is integrable as a function of s. Indeed, from assumption (A_1) it follows that $\int_{-\tau}^{0} g(s,\theta)\dot{x}^0(s+\theta)d\theta$ is a measurable function of s. Also due to (A_2) we obtain

$$\left| \int_{-\tau}^{0} g(s,\theta)\dot{x}^0(s+\theta)d\theta \right| \leq [\tfrac{1}{2}\, m(s)-|\mu|]\int_{-\tau}^{0} |\dot{x}^0(s+\theta)|\, d\theta$$

$$\leq [\tfrac{1}{2}\, m(s)-|\mu|]\int_{t_0-\tau}^{t_0+\beta} |\dot{x}^0(\theta)|\, d\theta .$$

Similar argument holds for all x^k s.

In the following we show that $\langle x^k \rangle$ is a Cauchy sequence. Now

$$(3.1.6)\qquad \| x^{k+1}-x^k\|_{[t_0-\tau,\, t_0+\beta]}$$

$$= \int_{t_0}^{t_0+\beta} |\dot{x}^{k+1}(t)-\dot{x}^k(t)|\, dt$$

$$\leq |\mu| \int_{t_0}^{t_0+\beta} |\dot{x}^k(t+c)-\dot{x}^{k-1}(t+c)|\, dt$$

$$+ \int_{t_0}^{t_0+\beta} (\tfrac{1}{2}\, m(t)-|\mu|)\left[\int_{-\tau}^{0} |\dot{x}^k(t+\theta)-\dot{x}^{k-1}(t+\theta)|\, d\theta\right]dt$$

$$\leq |\mu| \int_{t_0}^{t_0+\beta} \| x^k-x^{k-1}\|_{I_t}\, dt$$

$$+ \int_{t_0}^{t_0+\beta} (\tfrac{1}{2}\, m(t)-|\mu|)\, \| x^k-x^{k-1}\|_{I_t}\, dt$$

$$\leq \tfrac{1}{2} \int_{t_0}^{t_0+\beta} m(t)\, \|x^k-x^{k-1}\|_{I_t}\, dt.$$

Again, for $t \in [t_o, t_o+\beta]$,

(3.1.7) $$\| x^{k+1}-x^k\|_{I_t} = |x^{k+1}(t-\tau)-x^k(t-\tau)|+\int_{t-\tau}^{t} |\dot{x}^{k+1}(s)-\dot{x}^k(s)|\,ds$$

$$\leq 2|\mu| \int_{t_o}^{t} |x^k(s+c)-x^{k-1}(s+c)|\,ds$$

$$+2\int_{t_o}^{t}(\tfrac{1}{2}\, m(s)-|\mu|)[\int_{-\tau}^{0} |\dot{x}^k(s+\theta)-\dot{x}^{k-1}(s+\theta)|\,d\theta]\,ds$$

$$\leq \int_{t_o}^{t} m(s)\ \| x^k-x^{k-1}\|_{I_s}\ ds\ .$$

Further, for $t \in [t_o, t_o+\beta]$,

(3.1.8) $$\| x^1-\bar{\emptyset}\|_{I_t} \leq \int_{t_o}^{t} m(s)\ \|\bar{\emptyset}\|_{I_s}\,ds + \int_{t_o}^{t} m(s)\ ds$$

$$\leq (2\ \|\emptyset\|_{I_o} + 1)M(t)$$

Hence from (3.1.7) and (3.1.8) we obtain, for $t \in [t_o, t_o+\beta]$,

(3.1.9) $$\| x^{k+1}- x^k\|_{I_t} \leq (2\ \|\emptyset\|_{I_o} + 1)\ \frac{M^{k+1}(t)}{(k+1)!}$$

$$\leq (2\ \|\emptyset\|_{I_o} +1)\ \frac{M^{k+1}(t_o+\beta)}{(k+1)!}$$

Consequently, $\| x^{k+1}-x^k\|_{[t_o-\tau,t_o+\beta]} \leq \frac{1}{2}(2\|\emptyset\|_{I_o}+1)\ \dfrac{M^{k+1}(t_o+\beta)}{k!}$

This implies that $\langle x^k \rangle$ is a Cauchy sequence in $AC([t_o-\tau, t_o+\beta])$

Since the space $AC([t_o-\tau,t_o+\beta])$ is complete, there exists an x^* in $AC([t_o-\tau,t_o+\beta])$ such that

$$(3.1.10) \qquad \| x^k - x^* \|_{[t_o-\tau, t_o+\beta]} \to 0 \text{ as } k \to \infty .$$

From $\| x^k - x^* \|_{I_{t_o}} \leq \| x^k - x^* \|_{[t_o-\tau, t_o+\beta]}$ and (3.1.10) it follows that $x^*_{t_o} = \emptyset$.

Now since, for $t \in [t_o, t_o+\beta]$, $|x^k(t) - x^*(t)| \leq \| x^k - x^* \|_{[t_o-\tau, t_o+\beta]}$, we have from (3.1.10)

$$(3.1.11) \qquad \lim_{k \to \infty} x^k(t) = x^*(t) \ , \ t \in [t_o, t_o+\beta] .$$

Also, for $t \in [t_o, t_o+\beta]$,

$$\left| \int_{t_o}^{t} x^{k-1}(s+c)ds - \int_{t_o}^{t} x^*(s+c)\, ds \right|$$

$$\leq \int_{t_o}^{t} |x^{k-1}(s) - x^*(s)| ds$$

$$\leq \beta \, \|x^{k-1} - x^*\|_{[t_o-\tau, t_o+\beta]} ,$$

that is,

$$(3.1.12) \qquad \lim_{k \to \infty} \int_{t_o}^{t} x^{k-1}(s+c)ds = \int_{t_o}^{t} x^*(s+c)ds .$$

Further, for $t \in [t_o, t_o+\beta]$,

$$\left| \int_{t_o}^{t} \left[\int_{-\tau}^{0} g(s,\theta)\dot{x}^{k-1}(s+\theta)d\theta \right] ds - \int_{t_o}^{t} \left[\int_{-\tau}^{0} g(s,\theta)\dot{x}^*(s+\theta)d\theta \right] ds \right|$$

$$\leq \int_{t_o}^{t} (\tfrac{1}{2}\, m(s) - |\mu|) \left[\int_{-\tau}^{0} |\dot{x}^{k-1}(s+\theta) - \dot{x}^*(s+\theta)| d\theta \right] ds$$

$$\leq [\tfrac{1}{2}\, M(t_o+\beta) - \beta |\mu|] \, \| x^{k-1} - x^* \|_{[t_o-\tau, t_o+\beta]} .$$

Hence

$$(3.1.13) \quad \lim_{k\to\infty} \int_{t_0}^{t} [\int_{-\tau}^{0} g(s,\theta)\dot{x}^{k-1}(s+\theta)d\theta]ds$$

$$= \int_{t_0}^{t} [\int_{-\tau}^{0} g(s,\theta)\dot{x}^*(s+\theta)d\theta]\,ds$$

Passing to limit in (3.1.5) as $k \to \infty$ and making use of (3.1.11), (3.1.12) and (3.1.13) we obtain

$$x^*_{t_0} = \phi$$

$$x^*(t) = \phi(0) + \mu \int_{t_0}^{t} x^*(s+c)ds + \int_{t_0}^{t} [\int_{-\tau}^{0} g(s,\theta)\dot{x}^*(s+\theta)d\theta]ds$$

$$+ \int_{t_0}^{t} h(s)ds \ , \text{ for } t\in [t_0,t_0+\beta] \ .$$

This proves that x^* is a solution of (3.1.3) with initial function ϕ at t_0.

In what follows, we show that the solution x^* is unique. If possible let y^* be another solution of (3.1.3) existing in $[t_0,t_0+\beta]$ with the same initial function ϕ at t_0. Suppose there exists a t_1 in $[t_0,t_0+\beta]$ such that $x^*(t_1) \neq y^*(t_1)$. We set $|x^*(t_1)-y^*(t_1)| = c_1$, where $c_1>0$. Since $|x^*(t_1)-y^*(t_1)| \leq \| x^*-y^*\|_{[t_0-\tau,t_0+\beta]}$ then we can set $\|x^*-y^*\|_{[t_0-\tau,t_0+\beta]} = d$, where $d > 0$. Now

$$(3.1.14) \qquad \| x^*-y^*\|_{[t_0-\tau,t_0+\beta]} = \int_{t_0}^{t_0+\beta} |\dot{x}^*(s)-\dot{y}^*(s)|ds$$

$$\leq \frac{1}{2} \int_{t_0}^{t_0+\beta} m(s)\| x^*-y^*\|_{I_s} \, ds$$

and, for $t \in [t_o, t_o+\beta]$, $\| x^*-y^* \|_{I_t} \leq \int_{t_o}^{t} m(s) \| x^*-y^* \|_{I_s} ds$. Also, for $t \in [t_o, t_o+\beta]$,

$$\| x^*-y^* \|_{I_t} = |x^*(t-\tau)-y^*(t-\tau)| + \int_{t-\tau}^{t} |\dot{x}^*(s)-\dot{y}^*(s)| ds$$

$$\leq \| x^*-y^* \|_{[t_o-\tau, t_o+\beta]} + \int_{t_o-\tau}^{t_o+\beta} |\dot{x}^*(s)-\dot{y}^*(s)| ds$$

$$\leq 2 \| x^*-y^* \|_{[t_o-\tau, t_o+\beta]}$$

$$\leq 2d.$$

Hence $\| x^*-y^* \|_{I_t} \leq 2dM(t)$, $t \in [t_o, t_o+\beta]$. After n-repetitions, we obtain $\|x^*-y^*\|_{I_t} \leq 2d \dfrac{M^n(t_o+\beta)}{n!}$. Consequently from (3.1.14) it follows that $\| x^*-y^* \|_{[t_o-\tau, t_o+\beta]} \leq d \dfrac{M^{n+1}(t_o+\beta)}{n!}$. So passing to limit as $n \to \infty$, we get $\|x^*-y^*\|_{[t_o-\tau, t_o+\beta]} = 0$, that is, $|x^*(t_1)-y^*(t_1)| = 0$. This is a contradiction. Thus we complete the proof of the theorem.

COROLLARY 20. Since Theorem 19 holds for every $t_o+\beta$ in $[t_o, \infty)$, then we say that the system (3.1.3) admits a unique solution in $[t_o, \infty)$.

3.2 EXISTENCE OF SOLUTIONS OF ADJOINT SYSTEM

We consider the following integral equation:

$$(3.2.1) \qquad y(t) - \int_{t}^{\infty} y(s)\, g(s, t-s) ds + \mu \int_{t_o}^{t} y(s-c) ds = K,$$

where K is a n-row vector of constant elements, $t_0 \in (a,\infty)$ and $y:[t_0,\infty) \longrightarrow E^n$ and is such that the equation (3.2.1) is well-defined. This system is said to be adjoint to the linear homogeneous system (3.1.4).

<u>DEFINITION</u> 21. A function $y:[t_0,\infty) \longrightarrow E^n$ of bounded variation is said to be a solution of (3.2.1) in $[t_0,t_0+\alpha]$, $0<\alpha<\infty$, with an initial function $\psi:(t_0+\alpha,\infty) \longrightarrow E^n$ of bounded variation if $y(t) = \psi(t)$, for $t\in(t_0+\alpha,\infty)$ and satisfies the equation (3.2.1) for $t\in[t_0,t_0+\alpha]$.

Suppose that the assumption (A_5) holds. So, for $t\in[t_0,t_0+\alpha]$ with $t_0+\alpha+\tau<\infty$, $\int_{t_0+\alpha+\tau}^{\infty} y(s)g(s,t-s)ds = 0$. Consequently the system (3.2.1) takes the following form:

$$y(t) - \int_{t}^{t_0+\alpha+\tau} y(s)g(s,t-s)ds + \mu\int_{t_0}^{t} y(s-c)ds = K. \tag{3.2.2}$$

From this it follows that the problem above is still meaningful if ψ is known only in $(t_0+\alpha,t_0+\alpha+\tau]$.

<u>LEMMA</u> 22. Let X be a Banach space and B be a closed, bounded, convex subset of X. If $S:B \longrightarrow B$ is a contraction and $T:X \longrightarrow X$ is compact with $S(B)+T(B)\subset B$, then S+T has a fixed point in B. Here $S(B)+T(B) \overset{\text{def}}{=} \{u \mid u=Sy+Tz, y,z\in B\}$.

<u>THEOREM</u> 23. Suppose that the conditions (A_1), (A_3), (A_4), (A_6) and (A_7) are satisfied.

Then, for any $\psi\in BV((t_0+\alpha,t_0+\alpha+\tau])$, where $0<\alpha<\infty$ and $0\leq 2|\mu|\alpha<1$, the system (3.2.2) admits a solution in $[t_0,t_0+\alpha]$ with initial function ψ.

<u>Proof</u>: We define a function $M_1:[t_o-\tau,\infty) \longrightarrow R^+$ as follows:

$$M_1(t) = \begin{cases} 0 & , t\in I_{t_o} \\ \int_{t_o}^{t} m_1(s)ds, & t\in [t_o,\infty) \end{cases}$$

Let $X=BV([t_o,t_o+\alpha+\tau])$.

Let B be a set of functions $y\in X$ such that (i) $y(t)=\psi(t)$, for $t\in(t_o+\alpha,t_o+\alpha+\tau]$, (ii) $|y|_{[t_o,t_o+\alpha+\tau]} \leq L$, where

$$L \geq \frac{2|K| + 2K_1}{1-[3M^*(t_o,\alpha)+2|\mu|\alpha+\alpha N^*(t_o,\alpha)]}$$ and $K_1=|\psi|_{(t_o+\alpha,t_o+\alpha+\tau]}$ and

(iii) $|v(t_1)-v(t_2)| \leq \frac{1}{2} L|M_1(t_1)-M_1(t_2)|+LN^*(t_o,\alpha)|t_1-t_2|$, where $v=y|[t_o,t_o+\alpha]$.

In the following we show that B is a closed, bounded and convex set in X. Boundedness of B follows immediately from its definition. Let $y_o\in X$ be a limit point of B. So there exists a sequence $\langle y^k \rangle \subset B$ such that $|y^k-y^o|_{[t_o,t_o+\alpha+\tau]} \to 0$ as $k\to\infty$.

Since $\overset{t_o+\alpha+\tau}{\underset{t_o+\alpha}{V}} (y^k-y^o) \leq \overset{t_o+\alpha+\tau}{\underset{t_o}{V}} (y^k-y^o)$, then $\overset{t_o+\alpha+\tau}{\underset{t_o+\alpha}{V}} (\psi-y^o) = 0$, that is,

$\psi(t)-y^o(t) = C_o$, for $t\in(t_o+\alpha,t_o+\alpha+\tau]$, where C_o is a constant n-row vector. Further, since $|y^k(t_o+\alpha+\tau)-y^o(t_o+\alpha+\tau)| \leq |y^k-y^o|_{[t_o,t_o+\alpha+\tau]}$, then $\psi(t_o+\alpha+\tau) = y^o(t_o+\alpha+\tau)$. So $y^o(t) = \psi(t)$, for $t\in(t_o+\alpha,t_o+\alpha+\tau]$. Again $|y^o|_{[t_o,t_o+\alpha+\tau]} \leq |y^o-y^k|_{[t_o,t_o+\alpha+\tau]} + |y^k|_{[t_o,t_o+\alpha+\tau]}$ implies that $|y^o|_{[t_o,t_o+\alpha+\tau]} \leq L$.

Set $v^o = y^o|[t_o, t_o+\alpha]$ and $v^k = y^k|[t_o, t_o+\alpha]$. Now

$$|v^o(t_1)-v^o(t_2)| \le |v^o(t_1)-v^k(t_1)| + |v^k(t_1)-v^k(t_2)| + |v^k(t_2)-v^o(t_2)|$$

$$\le |y^o(t_1)-y^k(t_1)| + \frac{1}{2}L|M_1(t_1)-M_1(t_2)|$$

$$+ LN^*(t_o,\alpha)|t_1-t_2| + |y^k(t_2)-y^o(t_2)|$$

$$\le |y^o-y^k|_{[t_o,t_o+\alpha+\tau]} + \frac{1}{2}L|M_1(t_1)-M_1(t_2)|$$

$$+ LN^*(t_o,\alpha)|t_1-t_2| + |y^k-y^o|_{[t_o,t_o+\alpha+\tau]}.$$

So, in the limit as $k \to \infty$, we obtain

$$|v^o(t_1)-v^o(t_2)| \le \frac{1}{2}L|M_1(t_1)-M_1(t_2)| + LN^*(t_o,\alpha)|t_1-t_2| .$$

This proves that $y^o \in B$ and hence B is closed. For $y_1, y_2 \in B$, we set $z = \lambda y_1 + (1-\lambda)y_2$, where $0 \le \lambda \le 1$. It is easy to see that $z \in B$ and hence B is convex.

For $y \in X$, we define

$$(3.2.3) \qquad (Ty)(t) = \begin{cases} \displaystyle\int_t^{t_o+\alpha+\tau} y(s)g(s,t-s)ds, & t \in [t_o, t_o+\alpha] \\ 0 & , t \in (t_o+\alpha, t_o+\alpha+\tau], \end{cases}$$

and, for $y \in B$, we define

$$(3.2.4) \qquad (Sy)(t) = \begin{cases} K - \mu \displaystyle\int_{t_o}^{t} y(s-c)\, ds, & t \in [t_o, t_o+\alpha] \\ \psi(t) & , t \in (t_o+\alpha, t_o+\alpha+\tau] . \end{cases}$$

Clearly $T:X \longrightarrow X$. To show that T is continuous.

For $y_1, y_2 \in X$,

$$|Ty_1 - Ty_2|_{[t_o, t_o+\alpha+\tau]} = |Ty_1(t_o) - Ty_2(t_o)| + \overset{t_o+\alpha+\tau}{\underset{t_o}{V}} (Ty_1 - Ty_2) \ .$$

But $|Ty_1(t_o) - Ty_2(t_o)| \leq \int_{t_o}^{t_o+\alpha+\tau} |y_1(s) - y_2(s)|\,|g(s, t_o - s)|\,ds$

$$\leq |y_1 - y_2|_{[t_o, t_o+\alpha+\tau]} \int_{t_o}^{t_o+\alpha+\tau} \left(\tfrac{1}{2} m_1(s) - |\mu|\right) ds$$

and $\overset{t_o+\alpha+\tau}{\underset{t_o}{V}} (Ty_1 - Ty_2) \leq \overset{t_o+\alpha}{\underset{t_o}{V}} (Ty_1 - Ty_2) + |Ty_1(t_o+\alpha) - Ty_2(t_o+\alpha)|$

$$\leq |y_1 - y_2|_{[t_o, t_o+\alpha+\tau]} \int_{t_o}^{t_o+\alpha} \left(\tfrac{1}{2} m_1(s) - |\mu|\right) ds$$

$$+ |y_1 - y_2|_{[t_o, t_o+\alpha+\tau]}\, \alpha \int_{t_o}^{t_o+\alpha+\tau} N(s)\,ds$$

$$+ |y_1 - y_2|_{[t_o, t_o+\alpha+\tau]} \int_{t_o+\alpha}^{t_o+\alpha+\tau} \left(\tfrac{1}{2} m_1(s) - |\mu|\right) ds,$$

Since

$$\overset{t_o+\alpha}{\underset{t_o}{V}} (Ty_1 - Ty_2) = \sup_m \sum_{i=1}^{m} |(Ty_1(t_i) - Ty_2(t_i)) - (Ty_1(t_{i-1}) - Ty_2(t_{i-1}))|$$

$$\leq \sup_m \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} |y_1(s) - y_2(s)|\,|g(s, t_{i-1} - s)|\,ds$$

$$+ \sup_m \sum_{i=1}^{m} \int_{t_i}^{t_o+\alpha+\tau} |y_1(s) - y_2(s)|\,|g(s, t_i - s) - g(s, t_{i-1} - s)|\,ds$$

$$\leq |y_1-y_2|_{[t_o,t_o+\alpha+\tau]} \sup_m \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} (\tfrac{1}{2} m_1(s)-|\mu|)ds$$

$$+ |y_1-y_2|_{[t_o,t_o+\alpha+\tau]} \sup_m \sum_{i=1}^{m} \int_{t_o}^{t_o+\alpha+\tau} N(s)(t_i-t_{i-1})ds$$

$$\leq |y_1-y_2|_{[t_o,t_o+\alpha+\tau]} \int_{t_o}^{t_o+\alpha} (\tfrac{1}{2} m_1(s)-|\mu|)ds$$

$$+ |y_1-y_2|_{[t_o,t_o+\alpha+\tau]}\alpha \int_{t_o}^{t_o+\alpha+\tau} N(s)ds$$

Hence $|y_1-y_2|_{[t_o,t_o+\alpha+\tau]} \to 0$ implies that $|Ty_1-Ty_2|_{[t_o,t_o+\alpha+\tau]} \to 0$.

In the following we prove that $S(B)+T(B)\subset B$. Let $u\in S(B)+T(B)$. So

$$(3.2.5) \qquad u(t) = \begin{cases} K-\mu \int_{t_o}^{t} y(s-c)ds + \int_{t}^{t_o+\alpha+\tau} z(s)g(s,t-s)ds, \\ \qquad\qquad \text{for } t\in[t_o,t_o+\alpha] \\ \psi(t), \quad \text{for } t\in(t_o+\alpha,t_o+\alpha+\tau]. \end{cases}$$

Now

$$|u|_{[t_o,t_o+\alpha+\tau]} = |u(t_o)| + \overset{t_o+\alpha+\tau}{\underset{t_o}{V}} u$$

$$\leq |u(t_o)| + \overset{t_o+\alpha}{\underset{t_o}{V}} u + \overset{t_o+\alpha+\tau}{\underset{t_o+\alpha}{V}} u$$

$$+ |u(t_o+\alpha)| + |\psi|_{(t_o+\alpha,t_o+\alpha+\tau]}$$

$$\leq |u(t_o)|+|u(t_o+\alpha)|+ \overset{t_o+\alpha}{\underset{t_o}{V}} u+2|\psi|_{(t_o+\alpha,t_o+\alpha+\tau]}$$

$$\leq |u(t_o)|+|u(t_o+\alpha)| + \overset{t_o+\alpha}{\underset{t_o}{V}} u + 2K_1$$

But $|u(t_o)| \leq |K| + \int_{t_o}^{t_o+\alpha+\tau} |z(s)| \; |g(s,t_o-s)|ds$

$$\leq |K| + |z|_{[t_o,t_o+\alpha+\tau]} \int_{t_o}^{t_o+\alpha+\tau} \left(\frac{1}{2} m_1(s) - |\mu|\right)ds$$

$$\leq |K| + LM^*(t_o,\alpha)$$

and $|u(t_o+\alpha)| \leq |K|+|\mu| \int_{t_o}^{t_o+\alpha} |y(s-c)|ds + \int_{t_o+\alpha}^{t_o+\alpha+\tau} |z(s)||g(s,t_o+\alpha-s)|ds$

$$\leq |K| + \alpha|\mu| L + L M^*(t_o,\alpha)$$

Further, $\overset{t_o+\alpha}{\underset{t_o}{V}} u = \sup_m \sum_{i=1}^{m} |u(t_i)-u(t_{i-1})|$

$$\leq |\mu| \sup_m \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} |y(s-c)|ds$$

$$+ \sup_m \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} |z(s)||g(s,t_{i-1}-s)|ds$$

$$+ \sup_m \sum_{i=1}^{m} \int_{t_i}^{t_o+\alpha+\tau} |z(s)| \; |g(s,t_i-s)-g(s,t_{i-1}-s)|ds$$

$$\leq |\mu|\alpha|y|_{[t_o,t_o+\alpha+\tau]}+|z|_{[t_o,t_o+\alpha+\tau]}\int_{t_o}^{t_o+\alpha} \left(\frac{1}{2} m_1(s)-|\mu|\right)ds$$

$$+ |z|_{[t_o,t_o+\alpha+\tau]} \alpha \int_{t_o}^{t_o+\alpha+\tau} N(s)ds$$

$$\leq |\mu| \alpha L + L M^*(t_o,\alpha) + L \alpha N^*(t_o,\alpha)$$

So

$$|u|_{[t_o,t_o+\alpha+\tau]} \leq 2|K|+2K_1+3LM^*(t_o,\alpha)+2\alpha|\mu|L+L\alpha N^*(t_o,\alpha) \leq L.$$

Also, for $t_1, t_2 \in [t_o, t_o+\alpha]$,

$$|u(t_1)-u(t_2)| \leq |\mu|\left|\int_{t_1}^{t_2}|y(s-c)|ds\right| + \left|\int_{t_1}^{t_2}|z(s)||g(s,t_1-s)|ds\right|$$

$$+ \int_{t_2}^{t_o+\alpha+\tau} |z(s)||g(s,t_1-s)-g(s,t_2-s)|ds$$

$$\leq |\mu| \; |y|_{[t_o,t_o+\alpha+\tau]}|t_1-t_2|$$

$$+ |z|_{[t_o,t_o+\alpha+\tau]}\left|\int_{t_1}^{t_2}\left(\tfrac{1}{2}\, m_1(s)-|\mu|\right)ds\right|$$

$$+ |z|_{[t_o,t_o+\alpha+\tau]}\int_{t_o}^{t_o+\alpha+\tau} N(s)|t_1-t_2|ds$$

$$\leq |\mu| \; L|t_1-t_2|+L\left[\tfrac{1}{2}|M_1(t_1)-M_1(t_2)|-|\mu||t_1-t_2|\right]$$

$$+ LN^*(t_o,\alpha) \; |t_1-t_2|$$

$$\leq \tfrac{1}{2}L\,|M_1(t_1)-M_1(t_2)|+LN^*(t_o,\alpha)|t_1-t_2|.$$

Hence $u \in B$.

Proceeding as above it can be shown that $S(B) \subset B$, that is, $S: B \longrightarrow B$. Next to show that S is a contraction. Let $y_1, y_2 \in B$. Now

$$|Sy_1-Sy_2|_{[t_o,t_o+\alpha+\tau]}= |Sy_1(t_o)-Sy_2(t_o)|+ \overset{t_o+\alpha+\tau}{\underset{t_o}{V}} (Sy_1-Sy_2)$$

$$\leq \overset{t_o+\alpha}{\underset{t_o}{V}} (Sy_1-Sy_2)+|Sy_1(t_o+\alpha)-Sy_2(t_o+\alpha)|$$

$$\leq 2|\mu| \int_{t_o}^{t_o+\alpha} |y_1(s-c)-y_2(s-c)|ds$$

$$\leq 2|\mu| \alpha |y_1-y_2|_{[t_o,t_o+\alpha+\tau]}$$

So S is a contraction.

Now we prove that T is a compact mapping of B, that is, to show that T(B) is contained in a compact set in X. For this it is sufficient to prove that every sequence $\langle z^k \rangle$ in T(B) has a sub-sequence $\langle z^{k_i} \rangle$ which converges to a point in X. Let $G= \{ w| w=z|[t_o,t_o+\alpha],\ z\in T(B) \}$. From the assumptions on g it follows that $G\subset AC([t_o,t_o+\alpha])$. For $w\in G$,

$$\| w \|_{[t_o,t_o+\alpha]} = |w(t_o)| + \int_{t_o}^{t_o+\alpha} |\dot{w}(t)|dt$$

$$\leq \int_{t_o}^{t_o+\alpha+\tau} |y(s)||g(s,t_o-s)|ds$$

$$+ \int_{t_o}^{t_o+\alpha} |y(t)||g(t,0)|dt$$

$$+ \int_{t_o}^{t_o+\alpha} [\int_{t}^{t_o+\alpha+\tau} |y(s)|| \frac{\partial g(s,t-s)}{\partial t}|ds]\ dt$$

$$\leq |y|_{[t_o,t_o+\alpha+\tau]}\int_{t_o}^{t_o+\alpha+\tau} (\tfrac{1}{2}\ m_1(s)-|\mu|)ds$$

$$+ \; |y|_{[t_o, t_o+\alpha+\tau]} \int_{t_o}^{t_o+\alpha} \left(\tfrac{1}{2}\, m_1(t) - |\mu|\right) dt$$

$$+ \; |y|_{[t_o, t_o+\alpha+\tau]} \int_{t_o}^{t_o+\alpha} \left[\int_{t_o}^{t_o+\alpha+\tau} N(s)\,ds \right] dt$$

$$\leq L\,[2M^*(t_o,\alpha) + \alpha\, N^*(t_o,\alpha)] \;,$$

since $\dot{w}(t) = \int_t^{t_o+\alpha+\tau} y(s) \frac{\partial g(s,t-s)}{\partial t}\, ds - y(t)\; g(t,0)$. So G is uniformly bounded. Further,

$$\int_{t_o}^{t_o+\alpha} |\dot{w}(t+\sigma) - \dot{w}(t)|\, dt \leq \int_{t_o}^{t_o+\alpha} \left| \int_{t+\sigma}^{t_o+\alpha+\tau} y(s) \frac{\partial g(s,t+\sigma-s)}{\partial t} ds - \int_t^{t_o+\alpha+\tau} y(s) \frac{\partial g(s,t-s)}{\partial t} ds \right| dt$$

$$+ \int_{t_o}^{t_o+\alpha} |y(t+\sigma) g(t+\sigma,0) - y(t) g(t,0)|\, dt$$

$$\leq \int_{t_o}^{t_o+\alpha} \left[\int_{t+\sigma}^{t_o+\alpha+\tau} |y(s)| \left| \frac{\partial g(s,t+\sigma-s)}{\partial t} - \frac{\partial g(s,t-s)}{\partial t} \right| ds \right] dt$$

$$+ \int_{t_o}^{t_o+\alpha} \left| \int_t^{t+\sigma} |y(s)| \left| \frac{\partial g(s,t-s)}{\partial t} \right| ds \right| dt$$

$$+ \int_{t_o}^{t_o+\alpha} |y(t+\sigma) g(t+\sigma,0) - y(t+\sigma)\; g(t,0)|\, dt$$

$$+ \int_{t_o}^{t_o+\alpha} |y(t+\sigma)\; g(t,0) - y(t)\; g(t,0)|\, dt$$

$$\leq L \int_{t_o}^{t_o+\alpha} \left[\int_a^{t_o+\alpha+\tau} \left| \frac{\partial g(s,t+\sigma-s)}{\partial t} - \frac{\partial g(s,t-s)}{\partial t} \right| ds \right] dt$$

$$+ L \int_{t_o}^{t_o+\alpha} \left| \int_t^{t+\sigma} N(s)\,ds \right| dt$$

$$+ L \int_{t_o}^{t_o+\alpha} |g(t+\sigma,0) - g(t,0)|dt$$

$$+ \int_{t_o}^{t_o+\alpha} (\tfrac{1}{2} m_1(t)-|\mu|)|y(t+\sigma)-y(t)|dt$$

and $\int_{t_o}^{t_o+\alpha} (\tfrac{1}{2} m_1(t)-|\mu|)|y(t+\sigma)-y(t)|dt$

$$= \int_{t_o}^{t_o+\alpha-\sigma} (\tfrac{1}{2} m_1(t)-|\mu|)|v(t+\sigma)-v(t)|dt$$

$$+ \int_{t_o+\alpha-\sigma}^{t_o+\alpha} (\tfrac{1}{2} m_1(t)-|\mu|)|\psi(t+\sigma) - v(t)|dt$$

$$\leq \int_{t_o}^{t_o+\alpha-\sigma} (\tfrac{1}{2} m_1(t)-|\mu|)[\tfrac{1}{2}L|M_1(t+\sigma)-M_1(t)|+LN^*(t_o,\alpha)\sigma]dt$$

$$+ |\psi|_{(t_o+\alpha,t_o+\alpha+\tau]} \int_{t_o+\alpha-\sigma}^{t_o+\alpha} (\tfrac{1}{2} m_1(t)-|\mu|)dt$$

$$+ L \int_{t_o+\alpha-\sigma}^{t_o+\alpha} (\tfrac{1}{2} m_1(t) - |\mu|)dt.$$

Hence from assumptions (A_6) and (A_7) it follows that

$$\lim_{\sigma\to 0} \int_{t_o}^{t_o+\alpha} |\dot{w}(t+\sigma) - \dot{w}(t)|dt=0 \text{ uniformly in } w.$$

Consequently by lemma 1 we conclude that G is relatively compact. Next we set $w^k=z^k|[t_o,t_o+\alpha]$, where $z^k\in T(B)$. So $\langle w^k\rangle\subset G$. Since G is relatively compact then the sequence $\langle w^k\rangle$ has a subsequence $\langle w^{k_i}\rangle$, say, such that

$$\|w^{k_i}-w^o\|_{[t_o,t_o+\alpha]} \to 0 \text{ as } i\to\infty, \tag{3.2.6}$$

where $w^o \in AC([t_o, t_o+\alpha])$. Corresponding to the subsequence $\langle w^{k_i} \rangle$ of $\langle w^k \rangle$ we consider the subsequence $\langle z^{k_i} \rangle$ of $\langle z^k \rangle$. We define $z^o : [t_o, t_o+\alpha+\tau] \longrightarrow E^n$ as follows:

$$z^o(t) = \begin{cases} w^o(t) & , t \in [t_o, t_o+\alpha] \\ 0 & , t \in (t_o+\alpha, t_o+\alpha+\tau] . \end{cases}$$

So $z^o \in X$. Again we see that

$$\begin{aligned} |z^{k_i} - z^o|_{[t_o, t_o+\alpha+\tau]} &= |z^{k_i}(t_o) - z^o(t_o)| + \mathop{V}_{t_o}^{t_o+\alpha+\tau} (z^{k_i} - z^o) \\ &\leq |w^{k_i}(t_o) - w^o(t_o)| + \mathop{V}_{t_o}^{t_o+\alpha} (w^{k_i} - w^o) \\ &\quad + |w^{k_i}(t_o+\alpha) - w^o(t_o+\alpha)| \\ &\leq |w^{k_i}(t_o) - w^o(t_o)| + \int_{t_o}^{t_o+\alpha} |\dot{w}^{k_i}(t) - \dot{w}^o(t)|\,dt \\ &\quad + \| w^{k_i} - w^o \|_{[t_o, t_o+\alpha]} \\ &\leq 2 \, \| w^{k_i} - w^o \|_{[t_o, t_o+\alpha]} . \end{aligned}$$

Hence from (3.2.6) we obtain

$$|z^{k_i} - z^o|_{[t_o, t_o+\alpha+\tau]} \to 0 \text{ as } i \to \infty .$$

This proves that T is a compact mapping of B.

From lemma 22 it follows that the operator S+T has a fixed point in B. Let it be y^o. So from (3.2.3) and (3.2.4) we obtain

$$y^{o}(t) = \begin{cases} K-\mu \int_{t_o}^{t} y^{o}(s-c)ds + \int_{t}^{t_o+\alpha+\tau} y^{o}(s)g(s,t-s)ds, & t\in[t_o,t_o+\alpha] \\ \psi(t) & , t\in(t_o+\alpha,t_o+\alpha+\tau] . \end{cases}$$

This completes the proof of the theorem.

3.3 REPRESENTATION OF SOLUTIONS

Let the assumptions (A_1) - (A_8) be satisfied. Let $x(t) \equiv x(t,t_o,\emptyset,h)$, for $t \geq t_o$, be a unique solution of (3.1.3) with initial function $\emptyset \in AC(I_o)$ at $t_o \in (a,\infty)$. Let $Y(s,t)$ be a n x n matrix the elements of which are defined for $t \geq t_o$ and $s\in[t_o,t]$ with values in R and are functions of bounded variation in s for each t. Let $Y(s,t) = 0$, for $s > t$ and $Y(t,t) = E$, where 0 and E are n x n zero and identity matrix respectively. Further, let $Y(s,t)$ be a matrix solution of

$$(3.3.1) \qquad y(s) - \int_{s}^{\infty} y(\theta)g(\theta,s-\theta)d\theta + \mu\int_{t_o}^{s} y(\theta-c)\, d\theta = K$$

for $s\in[t_o,t]$, where K is same as in (3.2.1), that is,

$$(3.3.2) \qquad Y(s,t)- \int_{s}^{\infty} Y(\theta,t)g(\theta,s-\theta)d\theta + \mu \int_{t_o}^{s} Y(\theta-c,t)d\theta = \overline{K} ,$$

where $\overline{K}$ is a n x n matrix with each row K. Since $Y(\theta,t)=0$ for $\theta > t$, then (3.3.2) reduces to

$$(3.3.3) \qquad Y(s,t) - \int_{s}^{t} Y(\theta,t)g(\theta,s-\theta)d\theta + \mu \int_{t_o}^{s} Y(\theta-c,t)d\theta = \overline{K},$$

for $t \geq t_o$ and $s\in[t_o,t]$.

Integrating by parts we obtain

$$(3.3.4) \qquad Y(t,t)x(t)-Y(t_o,t)x(t_o) = \int_{t_o}^{t} \{d_s\, Y(s,t)\}x(s)+\int_{t_o}^{t} Y(s,t)\dot{x}(s)ds$$

Now

$$(3.3.5)\quad \int_{t_o}^{t} Y(s,t)\dot{x}(s)ds$$

$$= \int_{t_o}^{t} Y(s,t)[\mu x(s+c)+ \int_{-\tau}^{0} g(s,\theta)\dot{x}(s+\theta)d\theta+h(s)]\, ds$$

$$= \mu\int_{t_o}^{t} Y(s,t)x(s+c)ds + \int_{t_o}^{t} Y(s,t)\, [\int_{-\tau}^{0} g(s,\theta)\dot{x}(s+\theta)d\theta]\, ds$$

$$+ \int_{t_o}^{t} Y(s,t)h(s)ds$$

and

$$(3.3.6)\quad \int_{t_o}^{t} Y(s,t)\, [\int_{-\tau}^{0} g(s,\theta)\dot{x}(s+\theta)d\theta]\, ds$$

$$= \int_{t_o}^{t} Y(s,t)\, [\int_{s-\tau}^{s} g(s,\theta-s)\dot{x}(\theta)d\theta]\, ds$$

$$= \int_{t_o}^{t} Y(s,t)\, [\int_{t_o-\tau}^{t} g(s,\theta-s)\dot{x}(\theta)d\theta]\, ds$$

$$= \int_{t_o-\tau}^{t} [\int_{t_o}^{t} Y(s,t)\, g(s,\theta-s)ds]\, \dot{x}(\theta)\, d\theta$$

$$= [\{\int_{t_o}^{t} Y(s,t)g(s,\theta-s)ds\} x(\theta)]_{t_o-\tau}^{t}$$

$$- \int_{t_o-\tau}^{t} d_\theta\, [\int_{t_o}^{t} Y(s,t)g(s,\theta-s)ds]\, x(\theta)$$

$$= - \int_{t_o-\tau}^{t_o} d_\theta\, [\int_{t_o}^{t} Y(s,t)g(s,\theta-s)ds]\, x(\theta)$$

$$- \int_{t_o}^{t} d_\theta\, [\int_{t_o}^{t} Y(s,t)\, g(s,\theta-s)ds]\, x(\theta)$$

$$= -\int_{t_0-\tau}^{t_0} d_\theta [\int_{t_0}^{t_0+\tau} Y(s,t)g(s,\theta-s)ds]\, x(\theta)$$

$$- \int_{t_0}^{t} d_\theta [\int_{\theta}^{t} Y(s,t)\ g(s,\theta-s)ds]\ x(\theta),$$

where we used the fact that $\int_{t_0-\tau}^{s-\tau} g(s,\theta-s)\dot{x}(\theta)d\theta = 0,$

$\int_{s}^{t} g(s,\theta-s)\ \dot{x}(\theta)d\theta = 0,\ \int_{t_0}^{t} Y(s,t)g(s,t-s)ds = 0,$

$\int_{t_0}^{t} Y(s,t)g(s,t_0-\tau-s)ds=0,\ \int_{t_0-\tau}^{t_0} d_\theta [\int_{t_0+\tau}^{t} Y(s,t)g(s,\theta-s)ds]\, x(\theta) = 0$

and $\int_{t_0}^{\theta} Y(s,t)g(s,\theta-s)ds = 0$. Further,

$$(3.3.7)\quad \int_{t_0}^{t} Y(s,t)x(s+c)ds = \int_{t_0+c}^{t+c} Y(\theta-c,t)\ x(\theta)d\theta$$

$$= \int_{t_0+c}^{t_0} Y(\theta-c,t)x(\theta)d\theta + \int_{t_0}^{t} Y(\theta-c,t)x(\theta)d\theta$$

$$- \int_{t+c}^{t} Y(\theta-c,t)x(\theta)d\theta$$

$$= \int_{t_0+c}^{t_0} Y(\theta-c,t)x(\theta)d\theta$$

$$+ \int_{t_0}^{t} d_\theta [\int_{t_0}^{\theta} Y(s-c,t)ds]\ x(\theta),$$

since $\int_{t+c}^{t} Y(\theta-c,t)x(\theta)d\theta=0$. Hence making use of (3.3.5),(3.3.6)

and (3.3.7), we obtain from (3.3.4) that

$$(3.3.8)\quad x(t) = Y(t_o,t)\,\emptyset(0) - \int_{t_o-\tau}^{t_o} d_\theta\,[\int_{t_o}^{t_o+\tau} Y(s,t)g(s,\theta-s)ds]\,x(\theta)$$

$$+ \mu\int_{t_o+c}^{t_o} Y(\theta-c,t)x(\theta)d\theta + \int_{t_o}^{t} Y(s,t)h(s)ds$$

$$+ \int_{t_o}^{t} d_\theta\,[Y(\theta,t)-\int_{\theta}^{t} Y(s,t)g(s,\theta-s)ds + \mu\int_{t_o}^{\theta} Y(s-c,t)ds]\,x(\theta).$$

With the help of (3.3.3), we can write (3.3.8) as follows:

$$(3.3.9)\quad x(t) = Y(t_o,t)\emptyset(0) - \int_{t_o-\tau}^{t_o} d_\theta\,[\int_{t_o}^{t_o+\tau} Y(s,t)g(s,\theta-s)ds]\,\emptyset(\theta-t_o)$$

$$+ \mu\int_{t_o+c}^{t_o} Y(\theta-c,t)\emptyset(\theta-t_o)d\theta + \int_{t_o}^{t} Y(s,t)h(s)ds.$$

This is the required representation of the solution $x(t,t_o,\emptyset,h)$.

3.4 GENERAL ADJOINT SYSTEM TO LINEAR HOMOGENEOUS SYSTEM.

Let the assumptions $(A_1),(A_3)-(A_7)$ be satisfied. Let $x(t)=x(t,t_o,\emptyset)$, for $t\geq t_o$, be a unique solution of (3.1.4) with initial function $\emptyset\in AC(I_o)$ at $t_o\in(a,\infty)$. Let $y(t)$ be a solution of (3.2.1).

Now we consider only such values of $t>t_o$ such that $t>t_o+\tau$. We shall later remove this restriction. Integrating by parts we obtain

$$(3.4.1)\qquad y(t)x(t)-y(t_o)x(t_o) = \int_{t_o}^{t} (dy(s))x(s) + \int_{t_o}^{t} y(s)\dot{x}(s)ds$$

$$= \int_{t_o}^{t} (dy(s))x(s) + \mu\int_{t_o}^{t} y(s)x(s+c)ds$$

$$+ \int_{t_o}^{t} y(s) \; (\int_{-\tau}^{0} g(s,\theta)\dot{x}(s+\theta)d\theta) \; ds.$$

But

$$(3.4.2) \quad \int_{t_o}^{t} y(s)(\int_{-\tau}^{0} g(s,\theta)\dot{x}(s+\theta)d\theta)ds$$

$$= \int_{t_o}^{t} y(s)(\int_{s-\tau}^{s} g(s,\theta-s)\dot{x}(\theta)d\theta)ds$$

$$= \int_{t_o}^{t} y(s)(\int_{t_o-\tau}^{t} g(s,\theta-s)\dot{x}(\theta)d\theta)ds$$

$$= \int_{t_o-\tau}^{t} (\int_{t_o}^{t} y(s)g(s,\theta-s)ds)\dot{x}(\theta)d\theta$$

$$= [(\int_{t_o}^{t} y(s)g(s,\theta-s)ds)x(\theta)]_{t_o-\tau}^{t}$$

$$- \int_{t_o-\tau}^{t} d_\theta(\int_{t_o}^{t} y(s)g(s,\theta-s)ds) \; x(\theta)$$

$$= - \int_{t_o-\tau}^{t} d_\theta(\int_{t_o}^{t} y(s)g(s,\theta-s)ds)x(\theta)$$

$$- \int_{t_o}^{t-\tau} d_\theta(\int_{t_o}^{t} y(s)g(s,\theta-s)ds)x(\theta)$$

$$- \int_{t-\tau}^{t} d_\theta(\int_{t_o}^{t} y(s)g(s,\theta-s)ds)x(\theta)$$

$$= - \int_{t_o-\tau}^{t_o} d_\theta(\int_{t_o}^{t_o+\tau} y(s)g(s,\theta-s)ds)x(\theta)$$

$$- \int_{t_o}^{t-\tau} d_\theta(\int_{\theta}^{t} y(s)g(s,\theta-s)ds)x(\theta)$$

$$- \int_{t-\tau}^{t} d_\theta(\int_{\theta}^{t} y(s)g(s,\theta-s)ds)x(\theta) \; ,$$

since $\int_{t_0-\tau}^{s-\tau} g(s,\theta-s)\dot{x}(\theta)d\theta = 0$, $\int_{s}^{t} g(s,\theta-s)\dot{x}(\theta)d\theta = 0$,

$\int_{t_0}^{t} y(s)g(s,t-s)ds=0$, $\int_{t_0}^{t} y(s)g(s,t_0-\tau-s)ds=0$, $\int_{t_0}^{\theta} y(s)g(s,\theta-s)ds=0$

and $\int_{t_0-\tau}^{t_0} d_\theta(\int_{t_0+\tau}^{t} y(s)g(s,\theta-s)ds)x(\theta)=0$. Further since

$\int_{t_0}^{t-\tau} d_\theta(\int_{t}^{t+\tau} y(s)\ g(s,\theta-s)ds)x(\theta)=0$, we have by adding and subtracting

$\int_{t-\tau}^{t} d_\theta(\int_{t}^{t+\tau} y(s)g(s,\theta-s)ds)x(\theta)$ to right-hand side of (3.4.2)

$$(3.4.3)\quad \int_{t_0}^{t} y(s)(\int_{-\tau}^{0} g(s,\theta)\dot{x}(s+\theta)d\theta)ds$$

$$= - \int_{t_0-\tau}^{t_0} d_\theta(\int_{t_0}^{t_0+\tau} y(s)g(s,\theta-s)ds)x(\theta)$$

$$- \int_{t_0}^{t-\tau} d_\theta(\int_{\theta}^{t+\tau} y(s)g(s,\theta-s)ds)x(\theta)$$

$$- \int_{t-\tau}^{t} d_\theta (\int_{\theta}^{t+\tau} y(s)g(s,\theta-s)ds)x(\theta)$$

$$+ \int_{t-\tau}^{t} d_\theta(\int_{t}^{t+\tau} y(s)g(s,\theta-s)ds)x(\theta)$$

$$= - \int_{t_0-\tau}^{t_0} d_\theta(\int_{t_0}^{t_0+\tau} y(s)g(s,\theta-s)ds)x(\theta)$$

$$- \int_{t_0}^{t} d_\theta(\int_{\theta}^{t+\tau} y(s)g(s,\theta-s)ds)x(\theta)$$

$$+ \int_{t-\tau}^{t} d_\theta(\int_{t}^{t+\tau} y(s)g(s,\theta-s)ds)x(\theta) .$$

Again

$$(3.4.4)\quad \int_{t_o}^{t} y(s)x(s+c)ds = \int_{t_o+c}^{t+c} y(s-c)x(s)ds$$

$$= \int_{t_o+c}^{t_o} y(s-c)x(s)ds + \int_{t_o}^{t} y(s-c)x(s)ds - \int_{t+c}^{t} y(s-c)x(s)ds$$

$$= \int_{t_o+c}^{t_o} y(s-c)x(s)ds + \int_{t_o}^{t} d_s[\int_{t_o}^{s} y(\theta-c)d\theta]x(s) - \int_{t+c}^{t} y(s-c)\ x(s)ds$$

Using (3.4.3) and (3.4.4), we can write (3.4.1) as follows:

$$(3.4.5)\quad y(t)x(t) - \int_{t-\tau}^{t} d_\theta\left(\int_{t}^{t+\tau} y(s)g(s,\theta-s)ds\right)x(\theta) + \mu\int_{t+c}^{t} y(s-c)x(s)ds$$

$$= y(t_o)x(t_o) - \int_{t_o-\tau}^{t_o} d_\theta\left(\int_{t_o}^{t_o+\tau} y(s)g(s,\theta-s)ds\right)x(\theta) + \mu\int_{t_o+c}^{t_o} y(s-c)x(s)ds$$

$$+ \int_{t_o}^{t} d_\theta\,[y(\theta) - \int_{\theta}^{t+\tau} y(s)g(s,\theta-s)ds + \mu\int_{t_o}^{\theta} y(s-c)ds\,]\,x(\theta).$$

Since $\int_{t_o}^{t} d_\theta\left(\int_{t+\tau}^{\infty} y(s)g(s,\theta-s)ds\right)x(\theta)=0$, we obtain

$$\int_{t_o}^{t} d_\theta\,[y(\theta) - \int_{\theta}^{t+\tau} y(s)g(s,\theta-s)ds + \mu\int_{t_o}^{\theta} y(s-c)ds\,]\,x(\theta)$$

$$= \int_{t_o}^{t} d_\theta\,[y(\theta) - \int_{\theta}^{\infty} y(s)g(s,\theta-s)ds + \mu\int_{t_o}^{\theta} y(s-c)ds]\,x(\theta)$$

$$= 0\ ,$$

noting that $y(t)$ is a solution of (3.2.1). Hence (3.4.5) can be written as

$$(3.4.6)\qquad y(t)x(t)-\int_{t-\tau}^{t} d_\theta\Big(\int_{t}^{t+\tau} y(s)g(s,\theta-s)ds\Big)x(\theta)+\mu\int_{t+c}^{t} y(s-c)x(s)ds$$

$$= y(t_o)x(t_o)-\int_{t_o-\tau}^{t_o} d_\theta\Big(\int_{t_o}^{t_o+\tau} y(s)g(s,\theta-s)ds\Big)x(\theta)$$

$$+\mu\int_{t_o+c}^{t_o} y(s-c)x(s)ds,$$

for $t>t_o+\tau$, that is, $\mathbb{B}_t(x,y) = \mathbb{B}_{t_o}(x,y)$, for $t>t_o+\tau$, where $\mathbb{B}_t(x,y)$ denotes the left-hand side of (3.4.6). Since $\mathbb{B}_t(x,y)= \mathbb{B}_{t_o}(x,y)$ for t_o and t arbitrary such that $t>t_o+\tau$, then it is easy to show that $\mathbb{B}_t(x,y) = \mathbb{B}_{t_o}(x,y)$ for $t>t_o$. Hence we have

$$(3.4.7)\qquad y(t)x(t) - \int_{t-\tau}^{t} d_\theta\Big(\int_{t}^{t+\tau} y(s)g(s,\theta-s)ds\Big)x(\theta)$$

$$+\mu\int_{t+c}^{t} y(s-c)x(s)ds$$

$$=y(t_o)x(t_o)-\int_{t_o-\tau}^{t_o} d_\theta\Big(\int_{t_o}^{t_o+\tau} y(s)g(s,\theta-s)ds\Big)x(\theta)$$

$$+\mu\int_{t_o+c}^{t_o} y(s-c)x(s)ds,$$

for $t \geq t_o$.

CHAPTER 4

PERIODIC SOLUTIONS

This chapter deals with periodic neutral functional differential equations. We establish some theorems on the existence of periodic solutions of linear and quasi-linear systems by using fixed-point-theorem technique.

4.1 EXISTENCE AND UNIQUENESS OF SOLUTIONS.

We consider the following systems of linear and quasi-linear neutral differential equations:

$$\frac{dz}{dt} = f(t, z_t, \dot{z}_t) \tag{4.1.1}$$

$$\frac{dx}{dt} = f(t, x_t, \dot{x}_t) + h(t, x_t, \dot{x}_t, \mu) \tag{4.1.2}$$

where $f:(a,\infty) \times AC(I_0) \longrightarrow E^n$, $h:(a,\infty) \times AC(I_0) \times [-\mu_0, \mu_0] \longrightarrow E^n$, $\mu_0 > 0$ and $f(t, \psi, \dot{\psi})$ is linear in ψ. By an admissible function we mean an absolutely continuous function $x:[\eta_1 - \tau, \eta_2] \longrightarrow E^n$, where $a < \eta_1 < \eta_2 \leq \infty$, η_1 and η_2 may vary with x. Now suppose

(a_1) $f(t, x_t, \dot{x}_t)$ is measurable as a function of t for each fixed admissible x,

(a_2) $|f(t, \psi, \dot{\psi})| \leq K \|\psi\|_{I_0}$ for $t \in (a,\infty)$ and $\psi \in AC(I_0)$, where K is a positive constant,

(a_3) $h(t, x_t, \dot{x}_t, \mu)$ is measurable as a function of t for each fixed admissible x and $\mu \in [-\mu_0, \mu_0]$,

(a_4) $|h(t, \psi, \dot{\psi}, \mu) - h(t, \psi^*, \dot{\psi}^*, \mu)| \leq L(|\mu|) \, ||\psi - \psi^*||_{I_0}$,

for $t \in (a, \infty)$, $\psi, \psi^* \in AC(I_0)$ and $\mu \in [-\mu_0, \mu_0]$ where

$L: [0, \mu_0] \longrightarrow R^+$ is a continuous, non-decreasing function with $L(0) = 0$,

(a_5) $|h(t, \psi, \dot{\psi}, \mu_1) - h(t, \psi, \dot{\psi}, \mu_2)| \leq \eta(|\mu_1 - \mu_2|)$, for $t \in (a, \infty)$, $\psi \in AC(I_0)$ and $\mu_1, \mu_2 \in [-\mu_0, \mu_0]$, where $\eta: [0, 2\mu_0] \longrightarrow R^+$ is a continuous, non-decreasing function with $\eta(0) = 0$.

(a_6) $h(t, \psi, \dot{\psi}, 0) = 0$ for all $t \in (a, \infty)$ and $\psi \in AC(I_0)$.

<u>THEOREM</u> 24. Suppose that the assumptions (a_1)-(a_6) are satisfied. Then, for $t_0 \in (a, \infty)$, $\emptyset \in AC(I_0)$, and $\mu \in [-\mu_0, \mu_0]$, the system (4.1.2) admits a unique solution $x(t, t_0, \emptyset, \mu)$ existing for $t \in [t_0, t_0 + \beta]$, where $\beta > 0$.

<u>Proof</u>: We define a function $\bar{\emptyset}: [t_0 - \tau, \infty) \longrightarrow E^n$ as in (1.2.4), and a sequence $\langle x^k \rangle$ as follows:

$$x^0(t) = \bar{\emptyset}(t), \text{ for } t \in [t_0 - \tau, t_0 + \beta], \text{ and}$$

(4.1.3)
$$x^k_{t_0}(\theta) = \emptyset(\theta), \ \theta \in I_0$$

$$x^k(t) = \emptyset(0) + \int_{t_0}^{t} f(s, x_s^{k-1}, \dot{x}_s^{k-1})\, ds + \int_{t_0}^{t} h(s, x_s^{k-1}, \dot{x}_s^{k-1}, \mu)\, ds, \text{ for } t \in [t_0, t_0 + \beta],$$

where $k = 1, 2 \ldots$.

Clearly x^ks are well-defined. In the following we show that the sequence $\langle x^k \rangle$ is Cauchy. For this it is sufficient to show that

$$\|x^{k+1}-x^k\|_{[t_o-\tau,t_o+\beta]} \leq 4(K+L(|\mu|))^k\ (2K\|\emptyset\|_{I_o}+\eta(|\mu|))\ \frac{\beta^{k+1}}{k!}$$

Now

$$(4.1.4)\qquad \|x^{k+1}-x^k\|_{[t_o-\tau,t_o+\beta]}$$

$$= \int_{t_o}^{t_o+\beta} |\dot{x}^{k+1}(t) - \dot{x}^k(t)|\ dt$$

$$\leq \int_{t_o}^{t_o+\beta} |f(t,x_t^k,\dot{x}_t^k) - f(t,x_t^{k-1},\dot{x}_t^{k-1})|\,dt$$

$$+ \int_{t_o}^{t_o+\beta} |h(t,x_t^k,\dot{x}_t^k,\mu)-h(t,x_t^{k-1},\dot{x}_t^{k-1},\mu)|\,dt$$

$$\leq K \int_{t_o}^{t_o+\beta} \|x_t^k - x_t^{k-1}\|_{I_o}\,dt+L(|\mu|) \int_{t_o}^{t_o+\beta} \|x_t^k-x_t^{k-1}\|_{I_o}\,dt$$

$$\leq (K+L(|\mu|)) \int_{t_o}^{t_o+\beta} \|x_t^k-x_t^{k-1}\|_{I_o}\,dt.$$

An easy calculation shows that, for $t\in[t_o,t_o+\beta]$,

$$\|x^k-x^{k-1}\|_{I_t} \leq 2(K+L(|\mu|)) \int_{t_o}^{t} \|x^{k-1}-x^{k-2}\|_{I_s}\ ds$$

and $\|x^1-x^o\|_{I_t} \leq 2[2K\|\emptyset\|_{I_o} + \eta(|\mu|)]\,(t-t_o)$.

so, for $t\in[t_o,t_o+\beta]$,

$$(4.1.5)\qquad \|x^k-x^{k-1}\|_{I_t} \leq 4(K+L(|\mu|))^{k-1}(2K\|\emptyset\|_{I_o}+\eta(|\mu|))\ \frac{\beta^k}{k!}\ .$$

From (4.1.4) and (4.1.5) our assertion follows.

From the completeness of the space $AC([t_o-\tau, t_o+\beta])$, it follows that there exists an $x^* \in AC([t_o-\tau, t_o+\beta])$ such that

(4.1.6) $$\| x^k - x^* \|_{[t_o-\tau, t_o+\beta]} \to 0 \text{ as } k \to \infty.$$

Further, since, for $t \in [t_o, t_o+\beta]$, $|x^k(t)-x^*(t)| \leq \|x^k-x^*\|_{[t_o-\tau, t_o+\beta]}$ and $\|x^k-x^*\|_{I_t} \leq 2\|x^k-x^*\|_{[t_o-\tau, t_o+\beta]}$, then passing to limit in (4.1.3) as $k \to \infty$ we obtain

$$x^*_{t_o}(\theta) = \phi(\theta), \quad \theta \in I_o$$

$$x^*(t) = \phi(0) + \int_{t_o}^{t} f(s, x^*_s, \dot{x}^*_s)\,ds + \int_{t_o}^{t} h(s, x^*_s, \dot{x}^*_s, \mu)\,ds,$$

for $t \in [t_o, t_o+\beta]$.

This proves that $x^*(t)$ is a solution of (4.1.2)

To show that the solution $x^*(t)$ is unique we proceed exactly in the same way as in Theorem 4.

4.2 PERIODIC LINEAR SYSTEMS.

Let $f(t+\omega, \psi, \dot{\psi}) = f(t, \psi, \dot{\psi})$ for all $\psi \in AC(I_o)$ and $t \in (a, \infty)$, where $\tau < \omega$. Suppose that the assumptions (a_1) and (a_2) are satisfied. So, for $\phi \in AC(I_o)$ and $t_o \in (a, \infty)$, the system (4.1.1) admits a unique solution $z(t, t_o, \phi)$ existing for $t \in [t_o, t_o+\beta]$, where $\beta > 0$. We define a mapping $T: AC(I_o) \longrightarrow AC(I_o)$ by

$T\phi(\theta) = z_{t_o+\omega}(\theta, t_o, \phi)$, $\theta \in I_o$.

LEMMA 25. Suppose, for $t_0 \in (a,\infty)$

$$(a_7) \qquad \lim_{\sigma\to 0} \int_{t_0+\omega-\tau}^{t_0+\omega} |f(t+\sigma, z_{t+\sigma}, \dot{z}_{t+\sigma}) - f(t, z_t, \dot{z}_t)|\, dt = 0$$

uniformly with respect to admissible z. Then the operator T is completely continuous.

Proof: For, ϕ_1, $\phi_2 \in AC(I_0)$,

$$\| T\phi_1 - T\phi_2\|_{I_0} = \|z_{t_0+\omega}(t_0,\phi_1) - z_{t_0+\omega}(t_0,\phi_2)\|_{I_0}$$

$$\leq 2e^{2K\omega}\, \|\phi_1 - \phi_2\|_{I_0} .$$

This proves that the operator T is continuous.

In the following we show that T takes every bounded set in $AC(I_0)$ to a compact set in $AC(I_0)$. Let D^* be a bounded set in $AC(I_0)$ that is, $D^* = \{\psi \in AC(I_0) \mid \|\psi\|_{I_0} \leq M\}$. To prove that $T(D^*)$ is relatively compact. For $\psi \in T(D^*)$,

$$\|\psi\|_{I_0} = \|T\phi\|_{I_0} = \|z_{t_0+\omega}(t_0,\phi)\|_{I_0} \leq 2e^{2K\omega}\|\phi\|_{I_0} \leq 2Me^{2K\omega} .$$

So $T(D^*)$ is uniformly bounded. Further, since

$$\int_{-\tau}^{0} |\dot{\psi}(\theta+\sigma) - \dot{\psi}(\theta)|\, d\theta = \int_{-\tau}^{0} |\dot{z}_{t_0+\omega}(\theta+\sigma, t_0, \phi) - \dot{z}_{t_0+\omega}(\theta, t_0, \phi)|\, d\theta$$

$$= \int_{t_0+\omega-\tau}^{t_0+\omega} |\dot{z}(s+\sigma, t_0, \phi) - \dot{z}(s, t_0, \phi)|\, ds$$

$$= \int_{t_0+\omega-\tau}^{t_0+\omega} |f(s+\sigma, z_{s+\sigma}, \dot{z}_{s+\sigma}) - f(s, z_s, \dot{z}_s)|\, ds,$$

then from assumption (a_7) it follows that

$$\lim_{\sigma \to 0} \int_{-\tau}^{0} |\dot{\psi}(\theta+\sigma)-\dot{\psi}(\theta)|\,d\theta = 0$$

uniformly in ψ.

Hence from lemma 1 we conclude that $T(D^*)$ is relatively compact. Consequently the mapping T is completely continuous.

THEOREM 26. Suppose that the assumption (a_7) is satisfied. Let the system (4.1.1) admit a non-trivial solution $z(t,t_o,\phi^o)$ such that $\|z_{t_o+\omega}(t_o,\phi^o)-\phi^o\|_{I_o} \leq N_1$, where $0 < N_1 < \frac{N_1^*}{2}$ and $\|\phi^o\|_{I_o} = N_1^*$. Let the solutions of (4.1.1) satisfy the condition that $\|\phi-\phi^o\|_{I_o} \leq N_1$ implies $\|z_{t_o+\omega}(t_o,\phi) - \phi^o\|_{I_o} \leq N_1$, where $z(t,t_o,\phi)$ is any solution of (4.1.1) with initial function ϕ at t_o.

Then the system (4.1.1) admits a non-trivial periodic solution of period ω.

Proof: Let $B = \{\psi \in AC(I_o) \mid \|\psi-\phi^o\|_{I_o} \leq N_1\}$. Clearly $\psi \equiv 0$ does not belong to B and B is not empty. Moreover, B is a bounded, closed, convex set in $AC(I_o)$.

It is known that a solution $z(t,t_o,\phi)$ of (4.1.1) is periodic of period ω if and only if $z_{t_o+\omega}(t_o,\phi) = \phi$. We define a mapping $T: B \longrightarrow AC(I_o)$ by $T\phi(\theta)=z_{t_o+\omega}(\theta,t_o,\phi)$. From lemma 25 it follows that $T(B)$ is relatively compact. Clearly the mapping T is continuous. Next we show that $T(B) \subset B$. Let $\psi \in T(B)$. So there exists a $\phi \in B$ such that $\psi = T\phi$. Now

$$\|\psi-\phi^o\|_{I_o} = \|z_{t_o+\omega}(t_o,\phi)-\phi^o\|_{I_o} \leq N_1.$$

So $\psi \in B$. Consequently, from Theorem 14 it follows that T has a fixed point in B. Let it be ϕ^*. So $T\phi^* = \phi^*$, that is, $z_{t_0+\omega}(t_0,\phi^*) = \phi^*$. Hence the required periodic solution is $z(t,t_0,\phi^*)$. This completes the proof of the theorem.

4.3 PERIODIC QUASI-LINEAR SYSTEMS.

In this section we consider systems (4.1.1), (4.1.2) and (3.1.1) where $f(t,\psi,\dot{\psi})$ is linear in ψ. Let $f(t+\omega,\psi,\dot{\psi}) = f(t,\psi,\dot{\psi})$, $h(t+\omega,\psi,\dot{\psi},\mu) = h(t,\psi,\dot{\psi},\mu)$ and $h(t+\omega) = h(t)$ for all $t\in(a,\infty)$, $\mu \in [-\mu_0,\mu_0]$ and $\psi \in AC(I_0)$. Unique solutions of the above mentioned systems shall be denoted by $z(t,t_0,\phi)$, $x(t,t_0,\phi,\mu)$ and $x(t,t_0,\phi,h)$ respectively. So we can write

$$(4.3.1) \qquad x(t,t_0,\phi,h) = z(t,t_0,\phi) + \int_{t_0}^{t} Y(s,t)\, h(s)ds$$

and

$$(4.3.2) \qquad x(t,t_0,\phi,\mu) = z(t,t_0,\phi) + \int_{t_0}^{t} Y(s,t)h(s,x_s,\dot{x}_s,\mu)ds,$$

where $Y(s,t)$ is same as in 3.3.

<u>THEOREM</u> 27. Suppose that the assumption (a_7) is satisfied. The system (3.1.1) admits a periodic solution of period ω if and only if the system (4.1.1) does not have any periodic solution of period ω other than the trivial one.

<u>Proof</u>. From (4.3.1) we write

$$x_{t_0+\omega}(\theta,t_0,\phi,h) = z_{t_0+\omega}(\theta,t_0,\phi) + \int_{t_0}^{t_0+\omega+\theta} Y(s,t_0+\omega+\theta)h(s)ds .$$

we define an operator $T: AC(I_o) \longrightarrow AC(I_o)$ by $T\phi(\theta)=z_{t_o+\omega}(\theta,t_o,\phi)$. From lemma 25 it follows that the operator T is completely continuous. Now $x(t,t_o,\phi,h)$ is periodic of period ω if and only if $x_{t_o+\omega}(t_o,\phi,h) = \phi$, that is, if and only if

$$\phi(\theta) = T\phi(\theta) + \int_{t_o}^{t_o+\omega+\theta} Y(s,t_o+\omega+\theta)h(s)ds,$$

that is, if and only if,

$$(I - T)\ \phi(\theta) = \int_{t_o}^{t_o+\omega+\theta} Y(s,t_o+\omega+\theta)h(s)ds .$$

Since T is completely continuous, then $(I-T)^{-1}$ exists if and only if (4.1.1) has no ω -periodic solution other than the trivial one. This completes the proof of the theorem.

<u>THEOREM</u> 28. Suppose that the assumptions (a_1)-(a_7) are satisfied. Let the system (4.1.1) admit a non-trivial solution $z(t,t_o,\phi^o)$ such that $\|z_{t_o+\omega}(t_o,\phi^o) - \phi^o\|_{I_o} \leq \frac{1}{2} N_1$, where $0 < N_1 < \frac{1}{2} N_1^*$, and $\|\phi^o\|_{I_o} = N_1^*$. Let the solutions of (4.1.1) satisfy the condition that $\|\phi - \phi^o\|_{I_o} \leq N_1$ implies $\|z_{t_o+\omega}(t_o,\phi) - \phi^o\|_{I_o} \leq \frac{1}{2} N_1$. Further suppose that there exists an integrable function $J:(a,\infty) \longrightarrow R^+$ such that

$$|Y(s,t_1)-Y(s,t_2)| \leq J(s)|t_1-t_2|$$

for $s\in[t_o,\min\ (t_1,t_2)]$ and

$$\lim_{\sigma\to 0} \int_{t_o}^{t} \left| \frac{\partial}{\partial t} Y(s,t+\sigma) - \frac{\partial}{\partial t} Y(s,t)\right| ds = 0$$

uniformly in $t>t_o$.

Then, for sufficiently small μ in $[-\mu_0, \mu_0]$, the system (4.1.2) admits a periodic solution of period ω.

Proof: Let $U: AC(I_0) \to AC(I_0)$, $T: AC(I_0) \to AC(I_0)$ and $S: AC(I_0) \to AC(I_0)$ be defined by $U\phi(\theta) = x_{t_0+\omega}(\theta, t_0, \phi, \mu)$, $T\phi(\theta) = z_{t_0+\omega}(\theta, t_0, \phi)$ and

$$S\phi(\theta) = \int_{t_0}^{t_0+\omega+\theta} Y(s, t_0+\omega+\theta) h(s, x_s(t_0, \phi, \mu), \dot{x}_s(t_0, \phi, \mu), \mu) ds$$

respectively, where $t_0 \in (a, \infty)$ and $\mu \in [-\mu_0, \mu_0]$. From (4.3.2) it follows that $U = T+S$. Since it is known that a solution $x(t, t_0, \phi, \mu)$ of (4.1.2) is periodic of period ω if and only if $x_{t_0+\omega}(t_0, \phi, \mu) = \phi$, then it is sufficient to find a fixed point of the operator U, that is, of $T+S$.

From lemma 25 it follows that the operator T is completely continuous.

Let $B = \{\psi \in AC(I_0) \mid \|\psi - \phi^0\|_{I_0} \leq N_1\}$. Clearly $\psi \equiv 0$ does not belong to B and B is not empty. Moreover, B is a bounded, closed, convex set in $AC(I_0)$.

In the following we show that $(T+S)(B) \subset B$. Let $u \in (T+S)(B)$. So $u = T\phi + S\psi$, where ϕ and ψ are in B. Now $\|u - \phi^0\|_{I_0} \leq \|T\phi - \phi^0\|_{I_0} + \|S\psi\|_{I_0}$ and

$$\|S\psi\|_{I_0} = |S\psi(-\tau)| + \int_{-\tau}^{0} |\dot{S}\psi(\theta)| d\theta .$$

But $|S\psi(-\tau)| \leq \int_{t_0}^{t_0+\omega-\tau} |Y(s, t_0+\omega-\tau)| |h(s, x_s(t_0, \psi, \mu), \dot{x}_s(t_0, \psi, \mu), \mu)| ds$

$$\leq K_2 \int_{t_o}^{t_o+\omega-\tau} |h(s,x_s(t_o,\psi,\mu),\dot{x}_s(t_o,\psi,\mu),\mu)|\,ds$$

$$\leq K_2\,\omega\,\eta(|\mu|),$$

since $Y(s,t_o+\omega-\tau)$, $s\in[t_o,t_o+\omega-\tau]$ being bounded variation in s is bounded.

Further,

$$\int_{-\tau}^{0} |\dot{S}\psi(\theta)|\,d\theta \leq K \int_{t_o+\omega-\tau}^{t_o+\omega} \|x-z\|_{I_s}\,ds + \tau\eta(|\mu|)$$

and, for $t\in[t_o,t_o+\omega]$,

$$\|x-z\|_{I_t} \leq 2\,\omega\eta(|\mu|)e^{2K\omega}.$$

Hence $\|S\psi\|_{I_o} \leq [K_2\,\omega+2\omega K\,\tau\,e^{2K\omega}+\tau]\eta(|\mu|)$

$$\leq \tfrac{1}{2} N_1$$

This is possible for sufficiently small μ because $\eta(0)=0$ and η is continuous. Consequently,

$$\|u-\phi^o\|_{I_o} \leq \|z_{t_o+\omega}(t_o,\phi)-\phi^o\|_{I_o} + \|S\psi\|_{I_o}$$

$$\leq \tfrac{1}{2}N_1 + \tfrac{1}{2}N_1$$

$$\leq N_1.$$

So $u\in B$.

Next we show that S is a contraction. For $\phi_1,\phi_2\in C(I_o)$,

$$\|S\phi_1-S\phi_2\|_{I_o} = |S\phi_1(-\tau)-S\phi_2(-\tau)| + \int_{-\tau}^{0} |\dot{S}\phi_1(\theta)-\dot{S}\phi_2(\theta)|\,d\theta.$$

But $|S\phi_1(-\tau)-S\phi_2(-\tau)|$

$$\leq \int_{t_o}^{t_o+\omega-\tau} |Y(s,t_o+\omega-\tau)|\,|h(s,x_s(t_o,\phi_1,\mu),\dot{x}_s(t_o,\phi_1,\mu),\mu)$$

$$-h(s,x_s(t_o,\phi_2,\mu),\dot{x}_s(t_o,\phi_2,\mu),\mu)|\,ds$$

$$\leq K_2L(|\mu|)\int_{t_o}^{t_o+\omega} \|x_s(t_o,\phi_1,\mu)-x_s(t_o,\phi_2,\mu)\|_{I_o}\,ds$$

$$\leq 2\omega K_2L(|\mu|)\;e^{2(K+L(|\mu|))\omega}\;\|\phi_1-\phi_2\|_{I_o}$$

and

$$\int_{-\tau}^{0} |\dot{S}\phi_1(\theta)-\dot{S}\phi_2(\theta)|\,d\theta$$

$$\leq (t_o+\omega)L(|\mu|)\int_{-\tau}^{0}\{\int_{t_o}^{t_o+\omega+\theta} J(s)\|x(\phi_1)-x(\phi_2)\|_{I_s}\,ds\}\,d\theta$$

$$+\;(t_o+\omega)L(|\mu|)\int_{-\tau}^{0}\|x(\phi_1)-x(\phi_2)\|_{I_{t_o+\omega+\theta}}\,d\theta$$

$$\leq (t_o+\omega)L(|\mu|)\,[\int_{-\tau}^{0}\{\int_{t_o}^{t_o+\omega} J(s)\;\|x(\phi_1)-x(\phi_2)\|_{I_s}\,ds\}\,d\theta$$

$$+\int_{t_o}^{t_o+\omega}\|x(\phi_1)-x(\phi_2)\|_{I_s}\,ds]$$

$$\leq (t_o+\omega)L(|\mu|)[2J^{*}\tau\omega\;e^{2(K+L(|\mu|))\omega}\|\phi_1-\phi_2\|_{I_o}$$

$$+\;2\omega\,e^{2(K+L(|\mu|))\omega}\;\|\phi_1-\phi_2\|_{I_o}]$$

Consequently,

$$\|S\phi_1-S\phi_2\|_{I_0} \leq 2\omega L(|\mu|)e^{2(K+L(|\mu|))^{\omega}}[K_2+(t_0+\omega)(J^*\tau+1)]\,\|\phi_1-\phi_2\|_{I_0} .$$

Hence, for sufficiently small μ, S is a contraction. This follows from the properties of $L(|\mu|)$.

From lemma 22 it follows that T+S has a fixed point in B. Let it be ϕ^*. So the required periodic solution is $x(t,t_0,\phi^*,\mu)$. Further $\lim_{\mu\to 0} x(t,t_0,\phi^*\mu) = z(t,t_0,\phi^*)$.

This completes the proof of the theorem.

CHAPTER 5

STABILITY

In this chapter we establish theorems on uniform stability uniform asymptotic stability and exponential stability with the help of Liapumov functionals. Some converse theorems are also proved.

5.1 DEFINITIONS

We consider the following neutral functional differential equation:

$$\frac{dx}{dt} = f(t,x_t,\dot{x}_t), \tag{5.1.1}$$

where $f:R \times AC_H(I_o) \longrightarrow E^n$. Suppose that f satisfies conditions so that, for $t_o \in R$ and $\emptyset \in AC_H(I_o)$, the system (5.1.1) admits a unique solution $x(t,t_o,\emptyset)$ existing for $t \geq t_o$. So $x(\cdot,t_o,\emptyset)$: $[t_o-\tau,\infty) \longrightarrow E^n$ is an absolutely continuous function such that $x_t(t_o,\emptyset) \in AC_H(I_o)$ for $t \geq t_o$, $x_{t_o}(t_o,\emptyset) = \emptyset$ and $x(t,t_o,\emptyset)$ satisfies (5.1.1) for $t > t_o$.

In general $f(t,0,0) \neq 0$, that is, $x \equiv 0$ is not a solution of (5.1.1). But (5.1.1) can be transformed into another equation which has zero as a solution. Indeed, let $x^o(t)$ be a solution of (5.1.1). Setting $y=x-x^o$, we obtain

$$\frac{dy}{dt} = F(t,y_t,\dot{y}_t) , \tag{5.1.2}$$

where $F(t,y_t,\dot{y}_t) = f(t,y_t+x_t^o\ ,\ \dot{y}_t+\ \dot{x}_t^o\ ) - f(t,x_t^o,\ \dot{x}_t^o)$. Clearly $y \equiv 0$ is a solution of (5.1.2) and this corresponds to the solution $x^o(t)$ of (5.1.1). Hence without loss of generality we can assume that $x \equiv 0$ is a solution of (5.1.1)

DEFINITION 29. The trivial solution of (5.1.1) is stable, if for every $\varepsilon>0$ and $t_o \in R$, there exists a $\delta=\delta(\varepsilon,t_o)>0$ such that $\|\emptyset\|_{I_o} < \delta$ implies $\|x_t(t_o,\emptyset)\|_{I_o} < \varepsilon$ for $t \geq t_o$.

DEFINITION 30. The trivial solution of (5.1.1) is uniformly stable, if δ in Definition 29 is independent of t_o.

DEFINITION 31. The trivial solution of (5.1.1) is weakly uniformly stable, if for every $\varepsilon>0$ and $t_o \in R$, there exists a $\delta=\delta(\varepsilon)>0$ such that $\|\emptyset\|_{I_o} < \delta$ implies $|x(t,t_o,\emptyset)| < \varepsilon$ for $t \geq t_o$.

DEFINITION 32. The trivial solution of (5.1.1) is uniformly asymptotically stable, if it is uniformly stable and if given $\varepsilon>0$ and $t_o \in R$, there exists a $\delta_o>0$, independent of t_o and ε, and a $T=T(\varepsilon)>0$ such that $\|\emptyset\|_{I_o} < \delta_o$ implies $\|x_t(t_o,\emptyset)\|_{I_o} < \varepsilon$ for $t \geq t_o+T$.

DEFINITION 33. The trivial solution of (5.1.1) is weakly uniformly asymptotically stable, if it is weakly uniformly stable and if give $\varepsilon>0$ and $t_o \in R$, there exists a $\delta_o>0$, independent of t_o and ε, and a $T=T(\varepsilon)>0$ such that $\|\emptyset\|_{I_o} < \delta_o$ implies $|x(t,t_o,\emptyset)| < \varepsilon$ for $t \geq t_o+T$.

DEFINITION 34. The trivial solution of (5.1.1) is said to be exponentially stable if there exist constants B^* and α such that

$$\| x_t(t_o,\emptyset)\|_{I_o} \leq B^* \, e^{-\alpha(t-t_o)} \| \emptyset \|_{I_o}$$

for $t \geq t_o$.

DEFINITION 35. The trivial solution of (5.1.1) is said to be weakly exponentially stable if there exist constants B^* and α such that

$$|x(t,t_o,\emptyset)| \leq B^* \, e^{-\alpha(t-t_o)} \| \emptyset \|_{I_o}$$

for $t \geq t_o$.

Since $|x(t,t_o,\emptyset)| \leq \| x_t(t_o,\emptyset)\|_{I_o}$, for $t \geq t_o$, then from above definitions it follows that the uniform stability, uniform asymptotic stability and exponential stability of the trivial solution of (5.1.1) imply respectively the weak uniform stability, weak uniform asymptotic stability and weak exponential stability of the same.

5.2 UNIFORM STABILITY.

DEFINITION 36. A functional $V: R \times AC_H(I_o) \longrightarrow R^+$ is said to be continuous in $\psi \in AC_H(I_o)$ for each $t \in R$ if for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon, t) > 0$ such that $\| \psi - \psi^* \|_{I_o} < \delta$ implies $|V(t,\psi) - V(t,\psi^*)| < \varepsilon$.

THEOREM 37. Suppose that there exists a functional $V: R \times AC_H(I_o) \longrightarrow R^+$ such that

(i) $V(t,0) = 0$ and $V(t,\psi)$ is continuous in ψ for each t,

(ii) $a_1(|\psi(0)|) \leq V(t,\psi)$, where $a_1: [0,H] \longrightarrow R^+$ is a continuous, increasing and positive definite function,

(iii) $V^*(t) = V(t,x_t)$ is a decreasing function of t along any solution x of (5.1.1).

Then the trivial solution of (5.1.1) is weakly stable.

Proof: Given $0 < \varepsilon < H$. For a fixed $t_0 \in R$, we can choose a $\delta = \delta(\varepsilon, t_0) > 0$ such that $\|\psi\|_{I_0} < \delta$ implies $V(t_0, \psi) < a_1(\varepsilon)$. We claim that $\|\phi\|_{I_0} < \delta$ implies $|x(t,t_0,\phi)| < \varepsilon$ for $t \geq t_0$. If it is not true then there exists a $t_1 > t_0$ such that $|x(t_1,t_0,\phi)| = \varepsilon$. From property (iii) it follows that

$$V(t_1,x_{t_1}) = V^*(t_1) \leq V^*(t_0) = V(t_0,x_{t_0}) = V(t_0,\phi),$$

that is, $V(t_1,x_{t_1}) < a_1(\varepsilon)$. Further, using property (ii), we obtain

$$a_1(\varepsilon) = a_1(|x(t_1,t_0,\phi)|) \leq V(t_1,x_{t_1}) < a_1(\varepsilon).$$

This contradiction completes the proof of the theorem.

THEOREM 38. Suppose that there exists a functional $V: R \times AC_H(I_0) \longrightarrow R^+$ such that

(i) $a_1(|\psi(0)|) \leq V(t,\psi) \leq b_1(\|\psi\|_{I_0})$, where $a_1: [0,H] \longrightarrow R^+$ and $b_1: [0,H] \longrightarrow R^+$ are continuous, increasing and positive definite functions,

(ii) $V^*(t) = V(t,x_t)$ is a decreasing function of t along any solution x of (5.1.1).

Then the trivial solution of (5.1.1) is weakly uniformly stable.

Proof: Given $0 < \varepsilon < H$. From the properties on $b_1(r)$ it is possible to choose a $\delta = \delta(\varepsilon) > 0$ such that $b_1(\delta) < a_1(\varepsilon)$. We claim

that $|x(t,t_o,\emptyset)| < \varepsilon$, for $t \geq t_o$, whenever $\|\emptyset\|_{I_o} < \delta$. If possible let there exist a $t_1 > t_o$ such that $|x(t_1,t_o,\emptyset)| = \varepsilon$. So

$$a_1(\varepsilon) = a_1(|x(t_1,t_o,\emptyset)|) \leq V(t_1,x_{t_1}) = V^*(t_1) \leq V^*(t_o)$$

and

$$V^*(t_o) = V(t_o,x_{t_o}) = V(t_o,\emptyset) \leq b_1(\|\emptyset\|_{I_o}) < b_1(\delta) < a_1(\varepsilon).$$

Hence $a_1(\varepsilon) < a_1(\varepsilon)$.

From this contradiction the theorem follows.

THEOREM 39. If the trivial solution of (5.1.1) is weakly uniformly stable, then there exists a functional $V: R \times AC_H(I_o) \to R^+$ with the properties of Theorem 38.

Proof: Let $G(r)$, $r \geq 0$, be a continuous, increasing, positive definite real-valued function. Set

$$V(t,\psi) = \sup_{\sigma \geq 0} G(|x(t+\sigma,t,\psi)|) \tag{5.2.1}$$

for $t \in R$ and $\psi \in AC_H(I_o)$. So $V(t,\psi) \geq G(|\psi(0)|)$.

From the weak uniform stability of the trivial solution of (5.1.1) it follows that for every $\varepsilon > 0$ there exists a function $\delta = \delta(\varepsilon) > 0$ with the property that $\|\emptyset\|_{I_o} < \delta$ implies $|x(t,t_o,\emptyset)| < \varepsilon$ for $t \geq t_o$. The function $\delta(\varepsilon)$ may be choosen monotonic increasing and continuous. Hence there exists an inverse function $\varepsilon(\delta)$ with the property that $|x(t,t_o,\emptyset)| \leq \varepsilon(\|\emptyset\|_{I_o})$, for $t \geq t_o$, that is, $|x(t+\sigma,t,\psi)| \leq \varepsilon(\|\psi\|_{I_o})$ for $\sigma \geq 0$. So $V(t,\psi) \leq G(\varepsilon(\|\psi\|_{I_o}))$.

Further, for $t_1 \geq t_2$,

$$V^*(t_1) = V(t_1, x_{t_1})$$

$$= \sup_{\sigma \geq 0} G(|x(t_1+\sigma, t_1, x_{t_1}(t_o, \emptyset))|)$$

$$= \sup_{\sigma \geq 0} G(|x(t_1+\sigma, t_o, \emptyset)|)$$

$$= \sup_{\sigma \geq 0} G(|x(t_2+\sigma+\rho, t_o, \emptyset)|)$$

where $\rho = t_1 - t_2$. Hence

$$V^*(t_1) = \sup_{\sigma \geq \rho} G(|x(t_2+\sigma, t_o, \emptyset)|)$$

$$\leq \sup_{\sigma \geq 0} G(|x(t_2+\sigma, t_o, \emptyset)|)$$

$$\leq \sup_{\sigma \geq 0} G(|x(t_2+\sigma, t_2, x_{t_2}(t_o, \emptyset))|)$$

$$\leq V(t_2, x_{t_2})$$

$$\leq V^*(t_2) \ .$$

This completes the proof of the theorem.

Theorem on the uniform stability of the trivial solution of (5.1.1) may be obtained by replacing the property (i) of V function in Theorem 38 by

$$a_1(\|\psi\|_{I_o}) \leq V(t, \psi) \leq b_1(\|\psi\|_{I_o}).$$

Converse to theorem on uniform stability may also be obtained by setting

$$V(t, \psi) = \sup_{\sigma \geq 0} G(\| x_{t+\sigma}(t, \psi)\|_{I_o})$$

in Theorem 39.

5.3 UNIFORM ASYMPTOTIC STABILITY

DEFINITION 40. The upper right-hand derivative of a functional $V: R \times AC_H(I_o) \longrightarrow R^+$ along the solutions of (5.1.1) is denoted by $V'_{(5.1.1)}$ and is defined to be

$$(5.3.1) \quad V'_{(5.1.1)}(t, x_t(t_o, \emptyset)) = \overline{\lim_{\Delta t \to 0+}} \frac{1}{\Delta t} [V(t+\Delta t, x_{t+\Delta t}(t_o, \emptyset)) - V(t, x_t(t_o, \emptyset))] .$$

We shall also some times write

$$(5.3.2) \quad V'_{(5.1.1)}(t, \psi) = \overline{\lim_{\Delta t \to 0+}} \frac{1}{\Delta t} [V(t+\Delta t, x_{t+\Delta t}(t, \psi)) - V(t, x_t(t, \psi))] ,$$

where $x_t(t, \psi) = \psi$.

THEOREM 41. Suppose that there exists a functional $V: R \times AC_H(I_o) \longrightarrow R^+$ with the following properties:

(i) $a_1(|\psi(0)|) \leq V(t, \psi) \leq b_1(\|\psi\|_{I_o})$,

(ii) $V'_{(5.1.1)}(t, \psi) \leq - c_1(\|\psi\|_{I_o})$,

where $a_1: [0,H] \longrightarrow R^+$, $b_1: [0,H] \longrightarrow R^+$ and $c_1: [0,H] \longrightarrow R^+$ are continuous, increasing and positive definite functions.

Then the trivial solution of (5.1.1) is weakly uniformly asymptotically stable.

Proof: Set $V^*(t) = V(t, x_t(t_o, \emptyset))$. Since

$$\overline{\lim_{\Delta t \to 0+}} \frac{1}{\Delta t} [V^*(t+\Delta t) - V^*(t)] = \overline{\lim_{\Delta t \to 0+}} \frac{1}{\Delta t} [V(t+\Delta t, x_{t+\Delta t}(t_o, \emptyset)) - V(t, x_t(t_o, \emptyset))]$$

$$\leq - c_1(\|x_t(t_0,\phi)\|_{I_0})$$
$$< 0,$$

whenever $\phi \not\equiv 0$, then $V^*(t)$ is a decreasing function of t. Hence from Theorem 38 it follows that the trivial solution of (5.1.1) is weakly uniformly stable. Consequently, given $H > 0$ there exists a $\delta_0 = \delta_0(H) > 0$ such that $\|\phi\|_{I_0} < \delta_0$ implies $|x(t,t_0,\phi)| < H$ for $t \geq t_0$ and for $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that $\|\phi\|_{I_0} < \delta(\varepsilon)$ implies $|x(t,t_0,\phi)| < \varepsilon$, for $t \geq t_0$.

To show that, for the δ_0 determined above and for any $\eta > 0$, there exists a $T = T(\eta) > 0$ such that $\|\phi\|_{I_0} < \delta_0$ implies $|x(t,t_0,\phi)| < \eta$ for $t \geq t_0 + T$.

Set $T(\eta) = \dfrac{b_1(\delta_0)}{c_1(\delta(\eta))}$. If possible let $\|x_t(t_0,\phi)\|_{I_0} \geq \delta(\eta)$ for $t \in [t_0, t_0+T]$. So $c_1(\|x_t(t_0,\phi)\|_{I_0}) \geq c_1(\delta(\eta))$. Consequently $\overline{\lim\limits_{\Delta t \to 0+}} \frac{1}{\Delta t} [V^*(t+\Delta t) - V^*(t)] \leq - c_1(\delta(\eta))$. From this it follows that

$V^*(t) - V^*(t_0) \leq - c_1(\delta(\eta))\,(t-t_0)$, for $t \in [t_0, t_0+T]$, that is,

$$V(t, x_t(t_0,\phi)) \leq V(t_0,\phi) - c_1(\delta(\eta))\,(t-t_0)$$
$$\leq b_1(\|\phi\|_{I_0}) - c_1(\delta(\eta))\,(t-t_0)$$
$$< b_1(\delta_0) - c_1(\delta(\eta))\,(t-t_0).$$

So, for $t = t_0 + T$, we obtain $V(t_0+T, x_{t_0+T}(t_0,\phi)) < b_1(\delta_0) - c_1(\delta(\eta))T$, that is, $a_1(|x(t_0+T,t_0,\phi)|) < 0$. But this contradicts the hypothesis of the theorem. Hence there exists a t_1 in $[t_0, t_0+T]$

such that $\|x_{t_1}(t_o,\emptyset)\|_{I_o} < \delta(\eta)$. So from the weak uniform stability of the trivial solution of (5.1.1) it follows that $|x(t,t_1,x_{t_1}(t_o,\emptyset))| < \eta$ for $t \geq t_1$, that is, $|x(t,t_o,\emptyset)| < \eta$ for $t \geq t_1$. Thus we obtain $|x(t,t_o,\emptyset)| < \eta$ for $t \geq t_o+T$. This completes the proof of the theorem.

<u>THEOREM</u> 42. Suppose that there exists a functional $V: R \times AC_H(I_o) \longrightarrow R^+$ with the following properties:

(i) $\quad a_1(\|\psi\|_{I_o}) \leq V(t,\psi) \leq b_1(\|\psi\|_{I_o})$,

(ii) $\quad V'_{(5.1.1)}(t,\psi) \leq -c_1(\|\psi\|_{I_o})$,

where a_1, b_1 and c_1 are same as in Theorem 41.

Then the trivial solution of (5.1.1) is uniformly asymptotically stable.

The proof of this theorem is similar to that of Theorem 41.

<u>THEOREM</u> 43. If the trivial solution of (5.1.1) is uniformly asymptotically stable, then there exists a functional $V: R \times AC_H(I_o) \longrightarrow R^+$ which satisfies properties (i) and (ii) in Theorem 42.

<u>Proof</u>: Let $G(r)$, $r \geq 0$, be such that $G(0) = 0$, $G'(0) = 0$, $G(r) > 0$ and $G''(r) > 0$. Let $\alpha > 1$. Set $g(r) = G''(r)$. Then

$$G(r) = \int_0^r du \int_0^u g(v)dv \; , \; G\left(\frac{r}{\alpha}\right) = \int_0^{r/\alpha} du \int_0^u g(v)dv .$$

Putting $u = \frac{w}{\alpha}$, we obtain

$$G\left(\frac{r}{\alpha}\right) = \frac{1}{\alpha} \int_0^r dw \int_0^{w/\alpha} g(v)dv < \frac{1}{\alpha} \int_0^r dw \int_0^w g(v)dv = \frac{1}{\alpha} G(r).$$

Set, for $t \in R$ and $\psi \in AC_H(I_0)$,

$$V(t,\psi) = \sup_{\sigma \geq 0} G(\|x_{t+\sigma}(t,\psi)\|_{I_0}) \frac{1+\alpha\sigma}{1+\sigma} .$$

So $V(t,\psi) \geq G(\|\psi\|_{I_0})$. Since the trivial solution of (5.1.1) is uniformly stable then, proceeding as in Theorem 39, we obtain $V(t,\psi) \leq \alpha\, G(\varepsilon(\|\psi\|_{I_0}))$.

Since the trivial solution of the system (5.1.1) is uniformly asymptotically stable then $\|x_{t+\sigma}(t,\psi)\|_{I_0} < \eta$ for $\sigma \geq T(\eta)$. We may choose $T(\eta)$ to be a decreasing function in η. Putting $\eta = \frac{1}{\alpha}\|\psi\|_{I_0}$, we obtain $\|x_{t+\sigma}(t,\psi)\|_{I_0} < \frac{1}{\alpha}\|\psi\|_{I_0}$ for $\sigma \geq T(\frac{1}{\alpha}\|\psi\|_{I_0})$. Consequently, for $\sigma \geq T(\frac{1}{\alpha}\|\psi\|_{I_0})$,

$$\begin{aligned} G(\|x_{t+\sigma}(t,\psi)\|_{I_0}) \frac{1+\alpha\sigma}{1+\sigma} &< \alpha\, G(\tfrac{1}{\alpha}\|\psi\|_{I_0}) \\ &< G(\|\psi\|_{I_0}) \\ &< V(t,\psi). \end{aligned}$$

Hence

$$V(t,\psi) = \sup_{0 \leq \sigma \leq T(\frac{1}{\alpha}\|\psi\|_{I_0})} G(\|x_{t+\sigma}(t,\psi)\|_{I_0}) \frac{1+\alpha\sigma}{1+\sigma} .$$

Consequently,

$V(t+\Delta t, x_{t+\Delta t}(t,\psi))$

$$= \sup_{0 \leq \sigma \leq T(\frac{1}{\alpha}\|x_{t+\Delta t}(t,\psi)\|_{I_0})} G(\|x_{t+\Delta t+\sigma}(t+\Delta t, x_{t+\Delta t}(t,\psi))\|_{I_0}) \frac{1+\alpha\sigma}{1+\sigma}$$

$$= \sup_{0 \leq \sigma \leq T(\frac{1}{\alpha}\|x_{t+\Delta t}(t,\psi)\|_{I_0})} G(\|x_{t+\Delta t+\sigma}(t,\psi)\|_{I_0}) \frac{1+\alpha\sigma}{1+\sigma}$$

Since the function is continuous in σ, there exists a σ^* in $[0, T(\frac{1}{\alpha}\|x_{t+\Delta t}(t,\psi)\|_{I_0})]$ such that

$$V(t+\Delta t, x_{t+\Delta t}(t,\psi)) = G(\|x_{t+\Delta t+\sigma^*}(t,\psi)\|_{I_0})\frac{1+\alpha\sigma^*}{1+\sigma^*}$$

Setting $\sigma^*+\Delta t = \sigma$, we obtain

$$V(t+\Delta t, x_{t+\Delta t}(t,\psi)) = G(\|x_{t+\sigma}(t,\psi)\|_{I_0})\frac{1+\alpha\sigma^*}{1+\sigma^*}$$

$$= G(\|x_{t+\sigma}(t,\psi)\|_{I_0})\frac{1+\alpha\sigma}{1+\sigma}\left[1-\frac{(\alpha-1)\Delta t}{(1+\sigma^*)(1+\alpha\sigma)}\right],$$

because $\frac{1+\alpha\sigma}{1+\sigma}\left[1-\frac{(\alpha-1)\Delta t}{(1+\sigma^*)(1+\alpha\sigma)}\right] = \frac{1+\alpha\sigma}{1+\sigma} - \frac{\alpha\Delta t-\Delta t}{(1+\sigma)(1+\sigma^*)}$

$$= \frac{1+\alpha\sigma+\sigma^*+\alpha\sigma\sigma^*-\alpha\Delta t+\Delta t}{(1+\sigma)(1+\sigma^*)} = \frac{1+\sigma^*+\alpha\sigma\sigma^*+\alpha\sigma^*+\Delta t}{(1+\sigma)(1+\sigma^*)}$$

$$= \frac{1+\sigma+\alpha\sigma^*(1+\sigma)}{(1+\sigma)(1+\sigma^*)} = \frac{1+\alpha\sigma^*}{1+\sigma^*}.$$

So we have

$$V(t+\Delta t, x_{t+\Delta t}(t,\psi)) \le V(t,\psi)\left[1-\frac{(\alpha-1)\Delta t}{(1+\sigma^*)(1+\alpha\sigma)}\right],$$

that is,

$$\frac{1}{\Delta t}[V(t+\Delta t, x_{t+\Delta t}(t,\psi)) - V(t,\psi)] \le -\frac{(\alpha-1)V(t,\psi)}{(1+\sigma^*)(1+\alpha\sigma)},$$

where $0 \le \sigma^* \le T(\frac{1}{\alpha}\|x_{t+\Delta t}(t,\psi)\|_{I_0})$ and $\Delta t \le \sigma \le \Delta t + T(\frac{1}{\alpha}\|x_{t+\Delta t}(t,\psi)\|_{I_0})$.

Hence $\frac{1}{\Delta t}[V(t+\Delta t, x_{t+\Delta t}(t,\psi)) - V(t,\psi)]$

$$\le -\frac{(\alpha-1)V(t,\psi)}{[1+T(\frac{1}{\alpha}\|x_{t+\Delta t}(t,\psi)\|_{I_0})][1+\alpha\Delta t+\alpha T(\frac{1}{\alpha}\|x_{t+\Delta t}(t,\psi)\|_{I_0})]}$$

Since $T(\epsilon)$ is continuous and $V(t,\psi) \geq G(\|\psi\|_{I_o})$, then

$$V'_{(5.1.1)}(t,\psi) \leq - \frac{(\alpha-1)\; G(\|\psi\|_{I_o})}{[1+T(\frac{1}{\alpha}\|\psi\|_{I_o})][1+\alpha T(\frac{1}{\alpha}\|\psi\|_{I_o})]}$$

This completes the proof of the theorem.

5.4 EXPONENTIAL STABILITY.

In this section we consider

$$(5.4.1) \qquad \frac{dx}{dt} = f(t,x_t,\dot{x}_t) ,$$

where $f: R \times AC(I_o) \longrightarrow E^n$. Let $f(t,\psi,\dot{\psi})$ be linear in ψ. Suppose that f satisfies conditions so that the system (5.4.1) admits a unique solution $x(t,t_o,\emptyset)$ existing for $t \geq t_o$, where $t_o \in R$ and $\emptyset \in AC(I_o)$.

THEOREM 44. If the trivial solution of (5.4.1) is uniformly asymptotically stable, then it is exponentially stable.

Proof: We define an operator $U_{t,t_o}: AC(I_o) \longrightarrow AC(I_o)$ by $U_{t,t_o}\emptyset(\theta) = x_t(\theta,t_o,\emptyset)$ for $\theta \in I_o$ and $t \geq t_o$. The mapping U_{t,t_o} is linear due to the linearity of the system (5.4.1). Further

$$\|U_{t,t_o}\emptyset - U_{t,t_o}\psi\|_{I_o} = \|x_t(t_o,\emptyset) - x_t(t_o,\psi)\|_{I_o}$$
$$\leq 2\, e^{2L(t-t_o)} \|\emptyset - \psi\|_{I_o}$$

implies that U_{t,t_o} is continuous for fixed t and t_o, where L is the Lipschitz constant of the system (5.4.1). We define

$$\|U_{t,t_o}\| = \sup_{\|\psi\|_{I_o} \leq 1} \|U_{t,t_o}\psi\|_{I_o} .$$

So, for each fixed t and t_o, U_{t,t_o} is bounded.

From the uniform asymptotic stability of the trivial solution of (5.4.1) it follows that $\|x_t(t_o,\phi)\|_{I_o} < \varepsilon$, for $t \geq t_o+T$, whenever $\|\phi\|_{I_o} \leq \delta_o$, that is, $\|U_{t,t_o}\phi\|_{I_o} < \varepsilon$, for $t \geq t_o+T$, whenever $\|\phi\|_{I_o} \leq \delta_o$.

We fix ε, $0<\varepsilon<1$. Let ϕ^o be an arbitrary element of $AC(I_o)$ with $\|\phi^o\|_{I_o} \leq 1$. Consequently, $\|\delta_o\phi^o\|_{I_o} \leq \delta_o$. So $\|U_{t,t_o}\phi^o\|_{I_o} < \frac{\varepsilon}{\delta_o}$ and hence $\|U_{t,t_o}\| \leq \varepsilon/\delta_o$, for $t \geq t_o+T$, that is, $\|U_{t,t_o}\| \leq \varepsilon$, for $t \geq t_o+T_1$, where $T_1(\varepsilon) = T(\varepsilon\delta_o)$.

Since $x_t(\theta,t_o+T_1,x_{t_o+T_1}(t_o,\phi)) = x_t(\theta,t_o,\phi)$, then we have $U_{t,t_o}\phi = U_{t,t_o+T_1}(U_{t_o+T_1,t_o}\phi)$. So

$$\begin{aligned}\|U_{t,t_o}\phi\|_{I_o} &\leq \|U_{t,t_o+T_1}\| \, \|U_{t_o+T_1,t_o}\phi\|_{I_o} \\ &\leq \|U_{t,t_o+T_1}\| \, \|U_{t_o+T_1,t_o}\| \, \|\phi\|_{I_o} \\ &\leq \varepsilon^2 \|\phi\|_{I_o} ,\end{aligned}$$

for $t \geq t_o+2T_1$. By induction we obtain $\|U_{t,t_o}\phi\|_{I_o} \leq \varepsilon^m \|\phi\|_{I_o}$ for $t \geq t_o+mT_1$ and for all positive integers m. Hence $\|U_{t,t_o}\| \leq \varepsilon^m$ for $t \geq t_o+mT_1$.

In the following we show that $\|U_{t,t_o}\| < \frac{1}{\delta} e^{-\alpha(t-t_o)}$, for $t \geq t_o$. For any $t \geq t_o$, there exists an $m \geq 1$ such that $t_o+(m-1)T_1 \leq t < t_o+mT_1$. From the uniform stability of the trivial solution of (5.4.1), we obtain $\| U_{t,t_o} \| \leq \frac{1}{\delta} \varepsilon$, for $t \geq t_o$. We choose $0<\delta<\epsilon$ and set $\varepsilon = e^{-\alpha T_1}$ for some positive constant α. So $\|U_{t,t_o}\| \leq \frac{1}{\delta} e^{-\alpha T_1}$, for $t \geq t_o$. Consequently, for $t_o \leq t < t_o+T_1$, $\|U_{t,t_o}\| < \frac{1}{\delta} e^{-\alpha(t-t_o)}$. Further we have $\| U_{t,t_o} \| < \frac{\varepsilon^m}{\delta}$, for $t \geq t_o+(m-1)T_1$. So , for $m \geq 2$, $\|U_{t,t_o}\| < \frac{1}{\delta} \varepsilon^{\frac{t-t_o}{T_1}}$ when $t_o+(m-1)T_1 \leq t < t_o+mT_1$. This implies that, for $t \geq t_o+T_1$, $\|U_{t,t_o}\| < \frac{1}{\delta} e^{-\alpha(t-t_o)}$. Hence $\|U_{t,t_o}\| < \frac{1}{\delta} e^{-\alpha(t-t_o)}$, for $t \geq t_o$.

Setting $B^* = \frac{1}{\delta}$, we obtain $\|U_{t,t_o}\| < B^* e^{-\alpha(t-t_o)}$, for $t \geq t_o$. Hence $\|U_{t,t_o} \phi\|_{I_o} < B^* e^{-\alpha(t-t_o)} \| \phi \|_{I_o}$, $t \geq t_o$, that is,

$\|x_t(t_o,\phi)\|_{I_o} < B^* e^{-\alpha(t-t_o)} \| \phi\|_{I_o}$, for $t \geq t_o$.

This completes the proof of the theorem.

<u>THEOREM</u> 45. Suppose that there exists a continuous function $\alpha(t)$ possessing a continuous derivative for $t \geq 0$ and a positive continuous function $B^*(t)$, $t \geq 0$, such that for every $t_o \geq 0$, and $\phi \in AC(I_o)$ the solution $x(t,t_o,\phi)$ of (5.4.1) satisfies

(5.4.2) $$|x(t,t_o,\phi)| \leq B^*(t_o)\, e^{-[\alpha(t)-\alpha(t_o)]} \| \phi \|_{I_o} .$$

Then there exists a functional $V: R^+ \times AC(I_o) \longrightarrow R^+$ with the following properties:

(i) $|\psi(0)| \leq V(t,\psi) \leq B^*(t)\ \|\psi\|_{I_o}$

(ii) $|V(t,\psi_1) - V(t,\psi_2)| \leq B^*(t)\ \|\psi_1-\psi_2\|_{I_o}$,

(iii) $V'_{(5.1.4)}(t,\psi) \leq -\alpha'(t)\ V(t,\psi)$.

<u>Proof</u>: Let $V(t,\psi)$, for $t\in R^+$ and $\psi\in AC(I_o)$, be defined by

$$V(t,\psi) = \sup_{\sigma\geq 0} |x(t+\sigma,t,\psi)|\, e^{\alpha(t+\sigma)-\alpha(t)} .$$

So $V(t,\psi) \geq |\psi(0)|$ and $V(t,\psi) \leq B^*(t)\ \|\psi\|_{I_o}$. Further, for $\psi_1,\psi_2 \in AC(I_o)$, we obtain

$$|V(t,\psi_1) - V(t,\psi_2)|$$

$$= \left|\sup_{\sigma\geq 0} |x(t+\sigma,t,\psi_1)|\, e^{\alpha(t+\sigma)-\alpha(t)} - \sup_{\sigma\geq 0} |x(t+\sigma,t,\psi_2)|\, e^{\alpha(t+\sigma)-\alpha(t)}\right|$$

$$\leq \sup_{\sigma\geq 0} |x(t+\sigma,t,\psi_1)-x(t+\sigma,t,\psi_2)|\ e^{\alpha(t+\sigma)-\alpha(t)}$$

$$\leq \sup_{\sigma\geq 0} |x(t+\sigma,t,\psi_1-\psi_2)|\, e^{\alpha(t+\sigma)-\alpha(t)}$$

$$\leq B^*(t)\ \|\psi_1-\psi_2\|_{I_o} .$$

Also $V'_{(5.4.1)}(t,\psi)$

$$= \overline{\lim_{\Delta t\to 0+}} \frac{1}{\Delta t}[V(t+\Delta t, x_{t+\Delta t}(t,\psi)) - V(t,\psi)]$$

$$= \overline{\lim_{\Delta t\to 0+}} \frac{1}{\Delta t}\Big[\sup_{\sigma\geq 0} |x(t+\Delta t+\sigma, t+\Delta t, x_{t+\Delta t}(t,\psi))|\, e^{\alpha(t+\Delta t+\sigma)-\alpha(t+\Delta t)}$$

$$- \sup_{\sigma\geq 0} |x(t+\sigma,t,\psi)|\ e^{\alpha(t+\sigma)-\alpha(t)}\Big]$$

$$= \overline{\lim_{\Delta t \to 0+}} \frac{1}{\Delta t} [\sup_{\sigma \geq 0} |x(t+\Delta t+\sigma, t, \psi)| e^{\alpha(t+\Delta t+\sigma)-\alpha(t+\Delta t)}$$

$$- \sup_{\sigma \geq 0} |x(t+\sigma, t, \psi)| e^{\alpha(t+\sigma)-\alpha(t)}]$$

$$= \overline{\lim_{\Delta t \to 0+}} \frac{1}{\Delta t} [\sup_{\sigma \geq \Delta t} |x(t+\sigma, t, \psi)| e^{\alpha(t+\sigma)-\alpha(t)+\alpha(t)-\alpha(t+\Delta t)}$$

$$- \sup_{\sigma \geq 0} |x(t+\sigma, t, \psi)| e^{\alpha(t+\sigma)-\alpha(t)}]$$

$$\leq \overline{\lim_{\Delta t \to 0+}} \frac{1}{\Delta t} [\sup_{\sigma \geq 0} |x(t+\sigma, t, \psi)| e^{\alpha(t+\sigma)-\alpha(t)} \{e^{-(\alpha(t+\Delta t)-\alpha(t))} -1\}]$$

$$\leq V(t, \psi) \overline{\lim_{\Delta t \to 0+}} \frac{1}{\Delta t} \{e^{-(\alpha(t+\Delta t)-\alpha(t))} - 1\}$$

$$\leq - \alpha'(t) V(t, \psi),$$

That is, $V'_{(5.4.1)}(t, \psi) \leq - \alpha'(t) V(t, \psi)$.

This completes the proof of the theorem

COROLLARY. 46. Suppose that the solution $x(t, t_0, \emptyset)$ of (5.4.1) satisfies, in addition to condition (5.4.2) in Theorem 45, the following condition:

$$(5.4.3) \qquad \lim_{\eta \to 0} \int_{-\tau}^{0} |\dot{x}_{t+\eta}(\theta, t_0, \emptyset) - \dot{x}_t(\theta, t_0, \emptyset)| d\theta = 0$$

uniformly in $t \geq t_0$.

Then the functional $V(t, \psi)$ in Theorem 45 is continuous in (t, ψ).

Proof: Now

$$|V(t+\eta,\psi_1)-V(t,\psi_2)|\leq|V(t+\eta,\psi_1)-V(t+\eta,\psi_2)|$$

$$+|V(t+\eta,\psi_2)-V(t+\eta,x_{t+\eta}(t,\psi_2))|$$

$$+|V(t+\eta,x_{t+\eta}(t,\psi_2))-V(t,\psi_2)|.$$

But $|V(t+\eta,\psi_1)-V(t+\eta,\psi_2)| \leq B^*(t+\eta)\|\psi_1-\psi_2\|_{I_0}$

and $|V(t+\eta,\psi_2)-V(t+\eta,\ x_{t+\eta}(t,\psi_2))| \leq B^*(t+\eta)\|\psi_2- x_{t+\eta}(t,\psi_2)\|_{I_0}$,

where $\|\psi_2-x_{t+\eta}(t,\psi_2)\|_{I_0} = |\psi_2(-\tau)-x(t+\eta-\tau,t,\psi_2)|$

$$+\int_{-\tau}^{0}|\dot{\psi}_2(\theta)-\dot{x}_{t+\eta}(\theta,t,\psi_2)|\,d\theta.$$

Further,

$$|V(t+\eta,x_{t+\eta}(t,\psi_2))-V(t,\psi_2)|$$

$$=|\sup_{\sigma\geq 0}|x(t+\eta+\sigma,t,\psi_2)|e^{\alpha(t+\eta+\sigma)-\alpha(t+\eta)}-\sup_{\sigma\geq 0}|x(t+\sigma,t,\psi_2)|e^{\alpha(t+\sigma)-\alpha(t)}|$$

$$=|\sup_{\sigma\geq\eta}|x(t+\sigma,t,\psi_2)e^{\alpha(t+\sigma)-\alpha(t+\eta)}-\sup_{\sigma\geq 0}|x(t+\sigma,t,\psi_2)|e^{\alpha(t+\sigma)-\alpha(t)}|$$

Setting $u(\eta) = \sup_{\sigma\geq\eta}|x(t+\sigma,t,\psi_2)|e^{\alpha(t+\sigma)-\alpha(t)}$, we obtain

$$|V(t+\eta,x_{t+\eta}(t,\psi_2))-V(t,\psi_2)|=|u(\eta)e^{\alpha(t)-\alpha(t+\eta)}-u(0)|.$$

From the condition (5.4.2) it follows that $|x(t+\sigma,t,\psi_2)|e^{\alpha(t+\sigma)-\alpha(t)}$ is a bounded continuous function for all $\sigma\geq 0$. Hence $u(\eta)$ is a nonincreasing function of η with $u(\eta)\to u(0)$ as $\eta\to 0$.

Therefore $\eta\to 0$ and $\|\psi_1-\psi_2\|_{I_0}\to 0$ imply

$|V(t+n, \psi_1)-V(t,\psi_2)| \to 0.$

This proves our assertion.

THEOREM 47. Suppose that the condition (5.4.2) in Theorem 45 is replaced by

$$\|x_t(t_o,\phi)\|_{I_o} \leq B^*(t_o)e^{-[\alpha(t)-\alpha(t_o)]}\|\phi\|_{I_o} \tag{5.4.4}$$

where $B^*(t)$ and $\alpha(t)$ are same as in Theorem 45.

Then the conclusion of Theorem 45 follows with the property (i) replaced by

(i)' $\|\psi\|_{I_o} \leq V(t,\psi) \leq B^*(t)\ \|\psi\|_{I_o}$.

The proof is similar to that of Theorem 45 with

$$V(t, \psi) = \sup_{\sigma \geq 0} \|x_{t+\sigma}(t,\psi)\|_{I_o} e^{\alpha(t+\sigma)-\alpha(t)} .$$

COROLLARY 48. Suppose that in Theorem 47 $B^*(t) = B^*$, a positive constant, and $\alpha(t) = \alpha t$, where α is a positive constant. From Theorem 47 it follows that the exponential stability of the trivial solution of (5.4.1) implies the uniform asymptotic stability of the same.

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